6.10 The generalized Stokes's theorem

We worked hard to define the exterior derivative and to define orientation of manifolds and of boundaries. Now we are going to reap some rewards for our labor: we are going to see that there is a higher-dimensional analogue of the fundamental theorem of calculus, Stokes's theorem. It covers in one statement the four integral theorems of vector calculus, which are explored in section 6.11.

Recall the fundamental theorem of calculus:

Theorem 6.10.1 (Fundamental theorem of calculus). *If f is a C^1 function on a neighborhood of $[a,b]$, then*

$$\int_a^b f'(t)\,dt = f(b) - f(a). \qquad 6.10.1$$

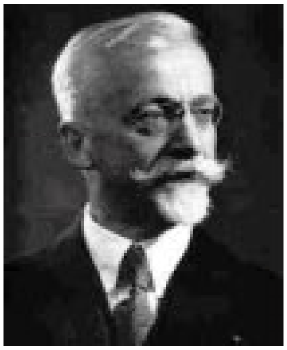

Figure 6.10.1.

Elie Cartan (1869–1951) formalized the theory of differential forms in the early twentieth century. Other names associated with the generalized Stokes's theorem include Henri Poincaré, Vito Volterra, and Luitzen Brouwer.

One of Cartan's four children, Henri, became a renowned mathematician; another, a physicist, was arrested by the Germans in 1942 and executed 15 months later.

Theorem 6.10.2 is probably the best tool mathematicians have for deducing global properties from local properties. It is a wonderful theorem.

It is often called the generalized Stokes's theorem, to distinguish it from the special case (surfaces in $\mathbb{R}^3$) also known as Stokes's theorem. Special cases of the generalized Stokes's theorem are discussed in section 6.11.

To lighten notation, in theorem 6.10.2 we write ∂X. However, we are actually integrating φ over $\partial^s_M X$, the smooth part of the boundary that sets off $X \subset M$ from M.

Restate this as

$$\int_{[a,b]} df = \int_{\partial[a,b]} f, \qquad 6.10.2$$

i.e., the integral of df over an oriented interval is equal to the integral of f over the oriented boundary of the interval. In this form, the statement generalizes to higher dimensions:

Theorem 6.10.2 (Generalized Stokes's theorem). *Let X be a compact piece-with-boundary of a $(k+1)$-dimensional oriented manifold $M \subset \mathbb{R}^n$. Give the boundary ∂X of X the boundary orientation, and let φ be a k-form defined on an open set containing X. Then*

$$\int_{\partial X} \varphi = \int_X d\varphi. \qquad 6.10.3$$

This beautiful, short statement is the main result of the theory of forms. Note that the dimensions in equation 6.10.3 make sense: if X is $(k+1)$-dimensional, ∂X is k-dimensional, and if φ is a k form, $d\varphi$ is a $(k+1)$-form, so $d\varphi$ can be integrated over X, and φ can be integrated over ∂X.

Example 6.10.3 (Integrating over the boundary of a square). You apply Stokes's theorem every time you use antiderivatives to compute an integral: to compute the integral of the 1-form $f\,dx$ over the oriented line segment $[a,b]$, you begin by finding a function g such that $dg = f\,dx$, and then say

$$\int_a^b f\,dx = \int_{[a,b]} dg = \int_{\partial[a,b]} g = g(b) - g(a). \qquad 6.10.4$$

This isn't quite the way Stokes's theorem is usually used in higher dimensions, where "looking for antiderivatives" has a different flavor.

For instance, to compute the integral $\int_C x\,dy - y\,dx$, where C is the boundary of the square S described by the inequalities $|x|, |y| \le 1$, with the boundary orientation, one possibility is to parametrize the four sides of the square (being careful to get the orientations right), then to integrate $x\,dy - y\,dx$ over all four sides and add. Another possibility is to apply Stokes's theorem:

$$\int_C x\,dy - y\,dx = \int_S d(x\,dy - y\,dx) = \int_S 2\,dx \wedge dy = \int_S 2|dx\,dy| = 8. \quad \triangle \qquad 6.10.5$$

The square S has sidelength 2, so its area is 4.

What is the integral over C of $x\,dy + y\,dx$? Check below.[22]

Example 6.10.4 (Integrating over the boundary of a cube). Let us integrate the 2-form

$$\varphi = (x - y^2 + z^3)\,(dy \wedge dz + dx \wedge dz + dx \wedge dy) \qquad 6.10.6$$

over the boundary of the cube C_a given by $0 \le x, y, z \le a$.

It is quite possible to do this directly, parametrizing all six faces of the cube, but Stokes's theorem simplifies things substantially.

Computing the exterior derivative of φ gives

$$\begin{aligned} d\varphi &= dx \wedge dy \wedge dz - 2y\,dy \wedge dx \wedge dz + 3z^2\,dz \wedge dx \wedge dy \\ &= (1 + 2y + 3z^2)\,dx \wedge dy \wedge dz, \end{aligned} \qquad 6.10.7$$

so we have

$$\begin{aligned} \int_{\partial C_a} \varphi &= \int_{C_a} (1 + 2y + 3z^2)\,dx \wedge dy \wedge dz \\ &= \int_0^a \int_0^a \int_0^a (1 + 2y + 3z^2)\,dx\,dy\,dz \\ &= a^2\big([x]_0^a + [y^2]_0^a + [z^3]_0^a\big) = a^2\big(a + a^2 + a^3\big). \quad \triangle \end{aligned} \qquad 6.10.8$$

Example 6.10.5 (Stokes's theorem: a harder example). Now let's try something similar but harder, integrating

$$\varphi = (x_1 - x_2^2 + x_3^3 - \cdots \pm x_n^n)\left(\sum_{i=1}^n dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right) \qquad 6.10.9$$

over the boundary of the n-dimensional cube C_a given by $0 \le x_j \le a$, for $j = 1, \dots, n$.

This time, the idea of computing the integral directly is pretty awesome: parametrizing all $2n$ faces of the cube, etc. Doing it using Stokes's theorem is also pretty awesome, but much more manageable. We know how to compute $d\varphi$, and it comes out to

$$d\varphi = \underbrace{(1 + 2x_2 + 3x_3^2 + \cdots + nx_n^{n-1})}_{\sum_{j=1}^n jx_j^{j-1}}\,dx_1 \wedge \cdots \wedge dx_n, \qquad 6.10.10$$

Example 6.10.5: Computing this exterior derivative is less daunting if you are alert for terms that can be discarded. Denote

$$(x_1 - x_2^2 + x_3^3 - \cdots \pm x_n^n)$$

by f. Then

$$\begin{aligned} D_1 f &= dx_1, \\ D_2 f &= -2x_2\,dx_2, \\ D_3 f &= 3x_3^2\,dx_3 \end{aligned}$$

and so on, ending with

$$\pm n x_n^{n-1}\,dx_n.$$

For $D_1 f$, the only term of

$$\sum_{i=1}^n dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

that survives is that in which $i = 1$, giving

$$dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

For $D_2 f$, the only term of the sum that survives is $dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n$, giving

$$-2x_2\,dx_2 \wedge dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n;$$

when the order is corrected this gives

$$2x_2\,dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

In the end, all the terms are followed simply by $dx_1 \wedge \cdots \wedge dx_n$, and any minus signs have become plus.

[22] $d(x\,dy + y\,dx) = dx \wedge dy + dy \wedge dx = 0$, so the integral is 0.