548 Chapter 5. Volumes of manifolds

a. Show that an alternative description of C is that it is the set of points that can be written in base 3 without using the digit 1. Use this to show that C is an uncountable set. (*Hint*: For instance, the number written as .02220000022202002222... in base 3 is in it.)

b. Show that C is a pavable set, with one-dimensional volume 0.

c. Show that the only dimension in which C can have volume different from 0 or infinity is $\ln 2 / \ln 3$.

5.5.2 This time let the set C be obtained from the unit interval by omitting the open middle 1/nth, then the open middle 1/n of each of the remaining intervals, then the open middle 1/nth of the remaining intervals, etc. (When n is even, this means omitting an open interval equivalent to 1/nth of the unit interval, leaving equal amounts on both sides, and so on.)

a. Show that C is a pavable set, with one-dimensional volume 0.

b. What is the only dimension in which C can have volume different from 0 or infinity? What is this dimension when n = 2?

5.6 Review Exercises for Chapter 5

5.1 Verify that equation 5.3.32 parametrizes the torus obtained by rotating around the z-axis the circle of radius r in the (x, z)-plane that is centered at x = R, z = 0.

5.2 Let $f : [a, b] \to \mathbb{R}$ be a smooth positive function. Find a parametrization for the surface of equation $\frac{x^2}{A^2} + \frac{y^2}{B^2} = \left(f(z)\right)^2$.

5.3 For what values of α does the spiral $\begin{pmatrix} r(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} 1/t^{\alpha} \\ t \end{pmatrix}, \alpha > 0$ between t = 1 and $t = \infty$ have finite length?

5.4 Compute the area of the graph of the function $f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{2}{3}(x^{3/2} + y^{3/2})$ above the region $0 \le x, y \le 1$.

5.5 Let f(x) be a positive C^1 function of $x \in [a, b]$.

a. Find a parametrization of the surface in \mathbb{R}^3 obtained by rotating the graph of f around the x-axis.

b. What is the area of this surface? (The answer should be in the form of a one-dimensional integral.)

5.6 Let $w_{n+1}(r) = \operatorname{vol}_{n+1}(B_r^{n+1}(\mathbf{0}))$ be the (n+1)-dimensional volume of the ball of radius r in \mathbb{R}^{n+1} , and let $v_n(r) = \operatorname{vol}_n(S_r^n)$ be the *n*-dimensional volume of the sphere of radius r in \mathbb{R}^{n+1} .

- a. Show that $w'_{n+1}(r) = v_n(r)$.
- b. Show that $v_n(r) = r^n v_n(1)$.
- c. Derive equation 5.3.54, using $w_{n+1}(1) = \int_0^1 w'_{n+1}(r) dr$.

5.6 Review Exercises for Chapter 5 549

The *total curvature* of a surface is defined in exercise 5.3.13.

5.7 Let *H* be the helicoid of equation $y \cos z = x \sin z$ (see example 3.8.12). What is the total curvature of the part of *H* with $0 \le z \le a$?

5.8 For
$$z \in \mathbb{C}$$
, the function $\cos z$ is by definition $\cos z = \frac{e^{iz} + e^{-iz}}{2}$.

a. If z = x + iy, write the real and imaginary parts of $\cos z$ in terms of x and y.

b. What is the area of the part of the graph of $\cos z$ where $-\pi \le x \le \pi$ and $-1 \le y \le 1$?

5.9 Let the set C be obtained from the unit interval [0, 1] by omitting the open middle 1/5th, then the open middle fifth of each remaining interval, then the open middle fifth of each remaining interval, etc.

a. Show that an alternative description of C is that it is the set of points that can be written in base 5 without using the digit 2. Use this to show that C is an uncountable set.

b. Show that C is a pavable set, with one-dimensional volume 0.

c. What is the only dimension in which C can have volume different from 0 or infinity?

5.10 Let $\vec{\mathbf{x}}_0, \vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_k$ be vectors in \mathbb{R}^n , and let $M \subset \mathbb{R}^n$ be the subspace spanned by $\vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_k$. Let G be the $k \times k$ matrix $G = [\vec{\mathbf{x}}_1 \dots \vec{\mathbf{x}}_k]^\top [\vec{\mathbf{x}}_1 \dots \vec{\mathbf{x}}_k]$ and let G^+ be the $(k+1) \times (k+1)$ matrix $G^+ = [\vec{\mathbf{x}}_0 \vec{\mathbf{x}}_1 \dots \vec{\mathbf{x}}_k]^\top [\vec{\mathbf{x}}_0 \vec{\mathbf{x}}_1 \dots \vec{\mathbf{x}}_k]$. The distance $d(\mathbf{x}, M)$ is by definition

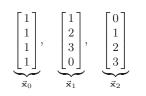
$$d(\mathbf{x}, M) = \inf_{\mathbf{y} \in M} (\mathbf{x} - \mathbf{y}).$$

a. Show that

$$\left(d(\mathbf{x}, M)\right)^2 = \frac{\det G^+}{\det G}$$

b. What is the distance between $\vec{\mathbf{x}}_0$ and the plane spanned by $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ (as defined in the margin)?

Exercise 5.10 is inspired by a proposition in *Functional Analysis, Vol 1: A Gentle Introduction*, by Dzung Minh Ha (Matrix Editions, 2006).



Vectors for exercise 5.10, part b