c. Parametrize the part of S where
i. $z > 0$; ii. $x > 0,\ y > 0$; iii. $z > x + y$.

Exercise 5.2.4, c, part iii is much harder than the others; even after finding an equation for the curve bounding the parametrizing region, you may need a computer to visualize it.

***5.2.5** Consider the open subset of $\mathbb{R}$ constructed in example 4.4.3: list the rationals between 0 and 1, say $a_1, a_2, a_3, \dots$, and take the union

$$U = \bigcup_{i=1}^{\infty} \left(a_i - \frac{1}{2^{i+k}}\ ,\ a_i + \frac{1}{2^{i+k}} \right)$$

for some integer $k > 2$. Show that U is a one-dimensional manifold and that it can be parametrized according to definition 5.2.3.

5.3 Computing volumes of manifolds

The k-dimensional volume of a k-dimensional manifold M embedded in any $\mathbb{R}^n$ is given by

$$\mathrm{vol}_k\, M = \int_M |d^k\mathbf{x}|, \tag{5.3.1}$$

where $|d^k\mathbf{x}|$ is the integrand that takes a k-parallelogram and returns its k-dimensional volume. Thus the length of a curve C can be written $\int_C |d^1\mathbf{x}|$, and the area of a surface S can be written $\int_S |d^2\mathbf{x}|$. Heuristically, this integral is defined by cutting up the manifold into little anchored k-parallelograms, adding their k-dimensional volumes and taking the limits of the sums as the decomposition becomes infinitely fine.

We know how to compute $|d^k\mathbf{x}|$ of a k-parallelogram: if $T = [\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]$, then

$$|d^k\mathbf{x}|P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \mathrm{vol}_k\, P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \sqrt{\det\,(T^\top T)}. \tag{5.3.2}$$

To compute the volume of a k-manifold M, we parametrize M by a mapping γ and then compute the volume of the k-parallelograms spanned by the *partial derivatives* of γ, sum them, and take the limit as the decomposition becomes infinitely fine. This gives the following definition.

In equation 5.3.3,

$$P_{\gamma(\mathbf{u})}\Big(\overrightarrow{D_1}\gamma(\mathbf{u}), \dots, \overrightarrow{D_k}\gamma(\mathbf{u})\Big)$$

is the k-parallelogram anchored at $\gamma(\mathbf{u})$ and spanned by the partial derivatives

$$\overrightarrow{D_1}\gamma(\mathbf{u}), \dots, \overrightarrow{D_k}\gamma(\mathbf{u});$$

$|d^k\mathbf{x}|$ of this parallelogram is the volume of the parallelogram (see proposition 5.1.1).

Definition 5.3.1 (Volume of manifold). Let $M \subset \mathbb{R}^n$ be a smooth k-dimensional manifold, U a pavable subset of $\mathbb{R}^k$, and $\gamma : U \to M$ a parametrization according to definition 5.2.3. Let X be as in that definition. Then

$$\begin{aligned} \mathrm{vol}_k\, M &= \int_{\gamma(U-X)} |d^k\mathbf{x}| \\ &= \int_{U-X} \Big(|d^k\mathbf{x}|\, \big(P_{\gamma(\mathbf{u})}\big(\overrightarrow{D_1}\gamma(\mathbf{u}), \dots, \overrightarrow{D_k}\gamma(\mathbf{u})\big)\big)\Big) |d^k\mathbf{u}| \\ &= \int_{U-X} \sqrt{\det([\mathbf{D}\gamma(\mathbf{u})]^\top [\mathbf{D}\gamma(\mathbf{u})])}\ |d^k\mathbf{u}|. \end{aligned} \tag{5.3.3}$$

Remark. When the manifold is a curve parametrized by $\gamma : [a,b] \to C$, equation 5.3.3 can be written

$$\int_C |d^1\mathbf{x}| = \int_{[a,b]} \sqrt{\det(\vec{\gamma}'(t) \cdot \vec{\gamma}'(t))}\, |dt| = \int_a^b |\vec{\gamma}'(t)|\, dt, \qquad 5.3.4$$

which is compatible with definition 3.8.5 of arc length. △

In definitions 5.3.1 and 5.3.2 we integrate over $U - X$, not U, because γ may not be differentiable on X. But X has k-dimensional volume 0, so this doesn't affect the integral.

Definition 5.3.1 is a special case of the following:

Definition 5.3.2 (Integral over a manifold with respect to volume). Let $M \subset \mathbb{R}^n$ be a smooth k-dimensional manifold, U a pavable subset of $\mathbb{R}^k$, and $\gamma : U \to M$ a parametrization, and let X be as in definition 5.2.3. Then $f : M \to \mathbb{R}$ is integrable over M with respect to volume if the integral on the right of equation 5.3.5 exists, and the integral is

$$\int_M f(\mathbf{x})|d^k\mathbf{x}| = \int_{U-X} f\big(\gamma(\mathbf{u})\big)\sqrt{\det([\mathbf{D}\gamma(\mathbf{u})]^\top [\mathbf{D}\gamma(\mathbf{u})])}\; |d^k\mathbf{u}|. \qquad 5.3.5$$

Definition 5.3.2: Such an integral is sometimes referred to as the *integral of a density*, as opposed to the integral of a differential form.

Let us see why definition 5.3.2 should be right. To simplify the discussion, let us consider the area of a surface parametrized by $\gamma : U \to \mathbb{R}^3$. This area should be

$$\lim_{N\to\infty} \sum_{C \in \mathcal{D}_N(\mathbb{R}^2)} \text{Area of } \gamma(C \cap U). \qquad 5.3.6$$

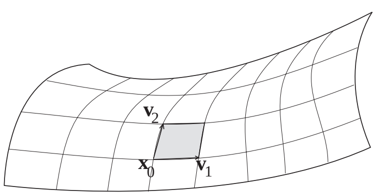

FIGURE 5.3.1.
A surface approximated by parallelograms. The point $\mathbf{x}_0$ corresponds to $\gamma(\mathbf{u})$, and the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ correspond to the vectors

$$\frac{1}{2^N}\overrightarrow{D_1}\gamma(\mathbf{u}) \quad \text{and} \quad \frac{1}{2^N}\overrightarrow{D_2}\gamma(\mathbf{u}).$$

That is, we make a dyadic decomposition of $\mathbb{R}^2$ and see how γ maps to S the dyadic squares C that are in U or straddle it. We then sum the areas of the resulting regions $\gamma(C \cap U)$. For $C \subset U$, this is the same as $\gamma(C)$; for C that straddle U, we add to the sum the area of the part of C that is in U.

The side length of a square C is $1/2^N$, so at least when $C \subset U$, the set $\gamma(C \cap U)$ is, as shown in figure 5.3.1, approximately the parallelogram

$$P_{\gamma(\mathbf{u})}\Big(\frac{1}{2^N}\overrightarrow{D_1}\gamma(\mathbf{u}), \frac{1}{2^N}\overrightarrow{D_2}\gamma(\mathbf{u})\Big), \qquad 5.3.7$$

where $\mathbf{u}$ is the lower left corner of C.

That parallelogram has area

$$\frac{1}{2^{2N}}\sqrt{\det[\mathbf{D}\gamma(\mathbf{u})]^\top [\mathbf{D}\gamma(\mathbf{u})]}. \qquad 5.3.8$$

So it seems reasonable to expect that the error we make by replacing

$$\text{Area of } \gamma(C \cap U) \quad \text{by} \quad \text{vol}_2(C)\sqrt{\det[\mathbf{D}\gamma(\mathbf{u})]^\top [\mathbf{D}\gamma(\mathbf{u})]} \qquad 5.3.9$$