5

Volumes of manifolds

5.0 Introduction

In chapter 4 we saw how to integrate over subsets of $\mathbb{R}^n$, first using dyadic pavings, then more general pavings. But these subsets were flat n-dimensional subsets of $\mathbb{R}^n$. What if we want to integrate over a (curvy) surface in $\mathbb{R}^3$?

When we say that in chapter 4 we had "flat domains" we mean we had n-dimensional subsets of $\mathbb{R}^n$. A disc in the plane is flat, even though its boundary is a circle: we cannot bend a disc and have it remain a subset of the plane. A subset of $\mathbb{R}$ is necessarily straight; if we want a wiggly line we must allow for at least two dimensions.

Many situations of obvious interest, like the area of a surface, or the total energy stored in the surface tension of a soap bubble, or the amount of fluid flowing through a pipe, are clearly some sort of surface integral. In a physics course, for example, you may have learned that the *electric flux* through a closed surface is proportional to the electric charge inside that surface.

In this chapter we will show how to compute the volume of a surface in $\mathbb{R}^3$, or more generally, a k-manifold in $\mathbb{R}^n$, where $k < n$. A first thing to realize is that we can't use the approach given in section 4.1, where we saw that when integrating over a subset $A \subset \mathbb{R}^n$,

$$\int_A g(\mathbf{x})\,|d^n\mathbf{x}| = \int_{\mathbb{R}^n} g(\mathbf{x})\mathbf{1}_A(\mathbf{x})\,|d^n\mathbf{x}|. \tag{4.1.8}$$

If we try to use this equation to integrate a function in $\mathbb{R}^3$ over a surface, the integral will certainly vanish, since the surface has three-dimensional volume 0. For any k-manifold M embedded in $\mathbb{R}^n$, with $k < n$, the integral would certainly vanish, since M has n-dimensional volume 0. Instead, we need to rethink the whole process of integration.

There is quite a bit of leeway when choosing what kind of "little pieces" to use; choosing a decomposition of a surface into little pieces is analogous to choosing a paving, and as we saw in section 4.7, there are many possible choices besides the dyadic paving.

At heart, integration is always the same:

> *Break up the domain into little pieces, assign a little number to each little piece, and finally add together all the numbers.* Then break the domain into littler pieces and repeat, taking the limit as the decomposition becomes infinitely fine. The *integrand* is the thing that assigns the number to the little piece of the domain.

In chapter 6 we will study a different kind of integrand, which assigns numbers to *oriented* manifolds.

The words "little piece" in this heuristic description need to be pinned down before we can do anything useful. We will chose to break the domain into k-dimensional parallelograms, and the "little number" we attach to each little parallelogram will be its k-dimensional volume. In section 5.1 we will see how to compute this volume.

We can only integrate over parametrized domains, and if we use the definition of parametrizations given in chapter 3, we will not be able to parametrize even such simple objects as the circle. Section 5.2 gives a looser definition of parametrization, sufficient for integration. In section 5.3 we compute volumes of k-manifolds; in section 5.4 we give an alternative description of curvature, in terms of the image of the *Gauss map*. Fractals and fractional dimension are discussed in section 5.5.

5.1 Parallelograms and their volumes

We saw in section 4.9 that the volume of a k-parallelogram in $\mathbb{R}^k$ is

$$\mathrm{vol}_k\, P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = |\det[\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]|. \qquad 5.1.1$$

What about a k-parallelogram in $\mathbb{R}^n$? Clearly if we draw a parallelogram on a rigid piece of cardboard, cut it out, and move it about in space, its area will not change. This area should depend only on the lengths of the vectors spanning the parallelogram and the angle between them; it should not depend on where they are placed in $\mathbb{R}^3$. But it isn't obvious how to compute this volume. Clearly equation 5.1.1 cannot be applied, as the determinant only exists for square matrices. A special formula (see proposition 1.4.19) exists for a 2-parallelogram in $\mathbb{R}^3$, but that formula is quite messy and requires the cross product. How will we compute the area of a 2-parallelogram in $\mathbb{R}^4$, where the cross product does not exist, never mind a 3-parallelogram in $\mathbb{R}^5$?

The following proposition is the key. It concerns k-parallelograms in $\mathbb{R}^k$, but we will be able to apply it to k-parallelograms in $\mathbb{R}^n$.

Proposition 5.1.1 (Volume of a k-parallelogram in $\mathbb{R}^k$). *Let $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ be k vectors in $\mathbb{R}^k$, so that $T = [\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]$ is a square $k \times k$ matrix. Then*

$$\mathrm{vol}_k\, P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \sqrt{\det(T^\top T)}. \qquad 5.1.2$$

Proof of proposition 5.1.1: Recall that if A and B are $n \times n$ matrices, then

$$\det A \det B = \det(AB)$$
$$\det A = \det A^\top$$

(Theorems 4.8.4 and 4.8.7).

Proof. $\sqrt{\det(T^\top T)} = \sqrt{(\det T^\top)(\det T)} = \sqrt{(\det T)^2} = |\det T| \quad \square$

Example 5.1.2 (Volume of two-dimensional and three-dimensional parallelograms). When $k = 2$, we have

$$\det(T^\top T) = \det\left(\begin{bmatrix} \vec{\mathbf{v}}_1^\top \\ \vec{\mathbf{v}}_2^\top \end{bmatrix} \begin{bmatrix} \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 \end{bmatrix}\right) = \det \begin{bmatrix} |\vec{\mathbf{v}}_1|^2 & \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 \\ \vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_1 & |\vec{\mathbf{v}}_2|^2 \end{bmatrix} \qquad 5.1.3$$
$$= |\vec{\mathbf{v}}_1|^2 |\vec{\mathbf{v}}_2|^2 - (\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2)^2.$$

Recall (definition 1.4.6) that

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = |\vec{\mathbf{x}}|\,|\vec{\mathbf{y}}| \cos\alpha,$$

where α is the angle between the vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$.

If we write $\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = |\vec{\mathbf{v}}_1||\vec{\mathbf{v}}_2| \cos\theta$ (where θ is the angle between $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$), this becomes

$$\det(T^\top T) = |\vec{\mathbf{v}}_1|^2 |\vec{\mathbf{v}}_2|^2 (1 - \cos^2\theta) = |\vec{\mathbf{v}}_1|^2 |\vec{\mathbf{v}}_2|^2 \sin^2\theta. \qquad 5.1.4$$

Thus proposition 5.1.1 asserts that the area of the 2-parallelogram spanned by $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ is

$$\sqrt{\det(T^\top T)} = |\vec{\mathbf{v}}_1||\vec{\mathbf{v}}_2||\sin\theta|, \qquad 5.1.5$$

which agrees with the formula of height times base given in high school: if $\vec{\mathbf{v}}_2$ is the base, then the height is $\vec{\mathbf{v}}_1 \sin\theta$.

Exactly the same computation in the case $k = 3$ leads to a much less familiar formula. Suppose $T = [\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$, and that the angle between $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$ is θ_1, the angle between $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_3$ is θ_2, and the angle between $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ is θ_3. Then

$$T^\top T = \begin{bmatrix} |\vec{\mathbf{v}}_1|^2 & \vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_2 & \vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_3 \\ \vec{\mathbf{v}}_2\cdot\vec{\mathbf{v}}_1 & |\vec{\mathbf{v}}_2|^2 & \vec{\mathbf{v}}_2\cdot\vec{\mathbf{v}}_3 \\ \vec{\mathbf{v}}_3\cdot\vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_3\cdot\vec{\mathbf{v}}_2 & |\vec{\mathbf{v}}_3|^2 \end{bmatrix} \tag{5.1.6}$$

and $\det T^\top T$ is given by

$$\begin{aligned} &|\vec{\mathbf{v}}_1|^2|\vec{\mathbf{v}}_2|^2|\vec{\mathbf{v}}_3|^2 + 2(\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_2)(\vec{\mathbf{v}}_2\cdot\vec{\mathbf{v}}_3)(\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_3) \\ &\qquad - |\vec{\mathbf{v}}_1|^2(\vec{\mathbf{v}}_2\cdot\vec{\mathbf{v}}_3)^2 - |\vec{\mathbf{v}}_2|^2(\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_3)^2 - |\vec{\mathbf{v}}_3|^2(\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_2)^2 \\ &= |\vec{\mathbf{v}}_1|^2|\vec{\mathbf{v}}_2|^2|\vec{\mathbf{v}}_3|^2\big(1 + 2\cos\theta_1\cos\theta_2\cos\theta_3 - (\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3)\big). \end{aligned} \tag{5.1.7}$$

For instance, the volume of a parallelepiped spanned by three unit vectors, each making an angle of $\pi/4$ with the others, is

$$\sqrt{1 + 2\cos^3\frac{\pi}{4} - 3\cos^2\frac{\pi}{4}} = \sqrt{\frac{\sqrt{2}-1}{2}}. \tag{5.1.8}$$

Note that we couldn't compute this volume using equation 5.1.1: to compute $|\det[\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]|$, we would need to know the entries of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. $\triangle$

Volume of a k-parallelogram in $\mathbb{R}^n$

The formula $\operatorname{vol}_k P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \sqrt{\det(T^\top T)}$ was useful in equation 5.1.8. But what really makes proposition 5.1.1 interesting is that the same formula can be used to compute the area of a k-parallelogram in $\mathbb{R}^n$.

Note that if T is an $n \times k$ matrix whose columns are $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$, then the product $T^\top T$ is a $k \times k$ matrix whose entries are all dot products of the vectors $\vec{\mathbf{v}}_i$:

$$\begin{array}{cc} & \overbrace{\begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \dots & \vec{\mathbf{v}}_k \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}}^{T} \\ \underbrace{\begin{bmatrix} \dots & \vec{\mathbf{v}}_1^\top & \dots \\ \dots & \vec{\mathbf{v}}_2^\top & \dots \\ \dots & \dots & \dots \\ \dots & \vec{\mathbf{v}}_k^\top & \dots \end{bmatrix}}_{T^\top} & \underbrace{\begin{bmatrix} |\vec{\mathbf{v}}_1|^2 & \vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_2 & \dots & \vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_k \\ \vec{\mathbf{v}}_2\cdot\vec{\mathbf{v}}_1 & |\vec{\mathbf{v}}_2|^2 & \dots & \vec{\mathbf{v}}_2\cdot\vec{\mathbf{v}}_k \\ \vdots & \vdots & \ddots & \dots \\ \vec{\mathbf{v}}_k\cdot\vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_k\cdot\vec{\mathbf{v}}_2 & \dots & |\vec{\mathbf{v}}_k|^2 \end{bmatrix}}_{T^\top T}. \end{array} \tag{5.1.9}$$

This matrix $T^\top T$ is identical to the matrix $T^\top T$ of equation 5.1.9: its entries can be computed from the lengths of the k vectors and the angles between them. No further information is needed. In particular, we do not need to know where the vectors are: at what point the parallelogram is anchored.

Thus we can use $\sqrt{\det(T^\top T)}$ to define k-dimensional volume in $\mathbb{R}^n$.

Definition 5.1.3 (Volume of a k-parallelogram in $\mathbb{R}^n$). Let the k vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ be in $\mathbb{R}^n$, and let T be the $n \times k$ matrix with these vectors as its columns: $T = [\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]$. Then the *$k$-dimensional volume* of $P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ is

$$\mathrm{vol}_k\, P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \sqrt{\det(T^\top T)}. \qquad 5.1.10$$

Exercise 5.1.3 asks you to show that if $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ are linearly dependent, $\mathrm{vol}_k(P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)) = 0$. In particular, this shows that if $k > n$, $\mathrm{vol}_k(P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)) = 0$.

Exercise 5.1.5 asks you to show that $\det(T^\top T) \geq 0$, so that definition 5.1.3 makes sense.

Example 5.1.4 (Volume of a 3-parallelogram in $\mathbb{R}^4$). What is the 3-dimensional volume of the 3-parallelogram P in $\mathbb{R}^4$ spanned by $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$? Set $T = [\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$; then

$$T^\top T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \det(T^\top T) = 4, \quad \text{so } \mathrm{vol}_3\, P = 2. \quad \triangle$$

Volume of anchored k-parallelograms

To break up a domain into little k-parallelograms we will need parallelograms "anchored" at different points in the domain. We denote by $P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ a k-parallelogram in $\mathbb{R}^n$ anchored at $\mathbf{x} \in \mathbb{R}^n$: the k vectors spanning the parallelogram all begin at $\mathbf{x}$. It is intuitively clear (and justified by proposition 4.1.21) that the vectors can be anchored at the origin or at any other point $\mathbf{x} \in \mathbb{R}^n$ without changing the volume of the parallelogram they span:

$$\mathrm{vol}_k\, P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \mathrm{vol}_k\, P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k). \qquad 5.1.11$$

The anchored k-parallelograms are the "little pieces" we will use when breaking up the domain.

The "little number" assigned to each piece will be its volume.

The need for parametrizations

Now we must address a more complex issue. The first step in integration is to "break up the domain into little pieces." In chapter 4 we had flat domains. Now we must break up a curvy domain into flat k-parallelograms.

For a curve, this is not hard. If $C \subset \mathbb{R}^n$ is a smooth curve, the integral $\int_C |d^1\mathbf{x}|$ is the number obtained by the following process: approximate C