FIGURE 4.11.1.
Henri Lebesgue (1875–1941)

After Lebesgue's father died of tuberculosis, leaving three children, the oldest five years old, his mother cleaned houses to support them. Lebesgue later wrote, "My first good fortune was to be born to intelligent parents, then to have been sickly and extremely poor, which kept me from violent games and distractions, ... and most of all to have an extraordinary mother even for France, this country of good mothers."

When one of Lebesgue's students, anxious and apprehensive, arrived for her first teaching job, replacing a popular substitute teacher, she found a note from him waiting for her. *Faites-vous aimer là-bas comme partout,* he had written ("make yourself beloved there as you are everywhere"). "It was a ray of sunshine," she recalled in a note published in *Message d'un mathématicien: Henri Lebesgue, pour le centenaire de sa naissance*, Paris, A. Blanchard, 1974.

$$f_k(x) = \begin{cases} 1 & \text{if } x \in \{a_1, \dots, a_k\} \\ 0 & \text{otherwise.} \end{cases} \tag{4.11.10}$$

Then

$$\int_0^1 f_k(x)\, dx = 0 \quad \text{for all } k, \tag{4.11.11}$$

but $\lim_{k\to\infty} f_k$ is the function that is 1 on the rationals and 0 on the irrationals between 0 and 1, and hence not integrable. △

The pitfalls of disappearing mass can be avoided by the dominated convergence theorem for Riemann integrals, theorem 4.11.4. In practice this theorem is not as useful as one might hope, because the hypothesis that the limit is Riemann integrable is rarely satisfied unless the convergence is uniform, in which case the much easier theorem 4.11.2 applies. But theorem 4.11.4 is the key tool in our approach to Lebesgue integration. The proof, in appendix A.21, is quite difficult and very tricky.

Theorem 4.11.4 (The dominated convergence theorem for Riemann integrals). *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be a sequence of integrable functions. Suppose that there exists R such that all f_k have their support in B_R, and all satisfy $|f_k| \le R$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function such that the set of $\mathbf{x}$ where $\lim_{k\to\infty} f_k(\mathbf{x}) \ne f(\mathbf{x})$ has measure 0. Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(\mathbf{x})\, |d^n\mathbf{x}| = \int_{\mathbb{R}^n} f(\mathbf{x})\, |d^n\mathbf{x}|.$$

Defining the Lebesgue integral

The weakness of theorem 4.11.4 is that we have to know that the limit is integrable. Usually we don't know this; most often, we need to deal with the limit of a sequence of functions, and all we know is that it is a limit. But we will now see that theorem 4.11.4 can be used to construct the Lebesgue integral, which is much better behaved under limits.

We abbreviate "Riemann integrable" as "R-integrable" and "Lebesgue-integrable" as "L-integrable." Proposition 4.11.5 is proved in appendix A.21.

Proposition 4.11.5 (Convergence except on a set of measure 0). *If f_k is a series of Riemann-integrable functions on $\mathbb{R}^n$ such that*

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\mathbf{x})||d^n\mathbf{x}| = A < \infty, \tag{4.11.12}$$

then $\sum_{k=1}^{\infty} f_k(\mathbf{x})$ converges except for $\mathbf{x}$ in a set X of measure 0.

Recall that "except on a set of measure 0" is also written "almost everywhere" (or a.e.). So above we could simply say that the sum converges almost everywhere.

We can now define "equal in the sense of Lebesgue," denoted $\underset{L}{=}$.

This notion of "Lebesgue equality" is fairly subtle, as sets of measure 0 can be quite complicated. For instance, if you only know a function almost everywhere, then you can never evaluate it at any point: you never know whether this is a point at which you know the function. The moral: functions that you know except on a set of measure 0 should only appear under integral signs.

Definition 4.11.6 (Lebesgue equality). Let f_k and g_k be two sequences of R-integrable functions such that

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\mathbf{x})||d^n\mathbf{x}| < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |g_k(\mathbf{x})||d^n\mathbf{x}| < \infty.$$

We will say that

$$\sum_{k=1}^{\infty} f_k \underset{L}{=} \sum_{k=1}^{\infty} g_k \qquad \text{if} \qquad \sum_{k=1}^{\infty} f_k(\mathbf{x}) = \sum_{k=1}^{\infty} g_k(\mathbf{x}) \quad \text{a.e.} \qquad 4.11.13$$

Theorem 4.11.7 will make it possible to define the Lebesgue integral.

Lebesgue integration is superior to Riemann integration. It does not require functions to be bounded with bounded support, it ignores "local nonsense" on sets of measure 0, and it is better behaved with respect to limits.

But if you want to compute integrals, the Riemann integral is still essential. Lebesgue integrals are more or less uncomputable unless you know a function as a limit of Riemann-integrable functions in an appropriate sense – in the sense of proposition 4.11.5, for instance.

Theorem 4.11.7. *Let f_k and g_k be two sequences of R-integrable functions such that*

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\mathbf{x})||d^n\mathbf{x}| < \infty, \qquad \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |g_k(\mathbf{x})||d^n\mathbf{x}| < \infty, \qquad 4.11.14$$

and

$$\sum_{k=1}^{\infty} f_k \underset{L}{=} \sum_{k=1}^{\infty} g_k. \qquad 4.11.15$$

Then

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(\mathbf{x})|d^n\mathbf{x}| = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} g_k(\mathbf{x})|d^n\mathbf{x}|. \qquad 4.11.16$$

Thus the integral of a function f that is the sum of a series of R-integrable functions as in 4.11.12 depends only on f and not on the series. So we can now define the Lebesgue integral.

Note that the series on the right of equation 4.11.18 is convergent, since (by part 4 of proposition 4.1.13) it is absolutely convergent:

$$\left| \int_{\mathbb{R}^n} f_k(\mathbf{x})|d^n\mathbf{x}| \right| \le \int_{\mathbb{R}^n} |f_k(\mathbf{x})||d^n\mathbf{x}| < \infty.$$

Definition 4.11.8 (Lebesgue integral). Let f_k be a sequence of R-integrable functions such that

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\mathbf{x})||d^n\mathbf{x}| < \infty. \qquad 4.11.17$$

Then the Lebesgue integral of $f = \sum_{k=1}^{\infty} f_k$ is

$$\int_{\mathbb{R}^n} f(\mathbf{x})|d^n\mathbf{x}| = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(\mathbf{x})|d^n\mathbf{x}|. \qquad 4.11.18$$