3.8.5 Check that if you consider the *surface* of equation $z = f(x)$, y arbitrary, and the plane *curve* $z = f(x)$, the mean curvature of the surface is half the curvature of the plane curve.

Useful fact for exercise 3.8.7: The arctic circle is those points that are 2 607.5 kilometers south of the north pole.

3.8.6 What are the Gaussian and mean curvature of the surface of equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \text{at} \begin{pmatrix} x \\ y \\ z \end{pmatrix}?$$

3.8.7 a. How long is the arctic circle? How long would a circle of that radius be if the earth were flat?

b. How big a circle around the pole would you need to measure in order for the difference of its length and the corresponding length in a plane to be one kilometer?

3.8.8 a. Draw the cycloid given parametrically by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a(t - \sin t) \\ a(1 - \cos t) \end{pmatrix}$.

b. Can you relate the name "cycloid" to "bicycle"?

c. Find the length of one arc of the cycloid.

3.8.9 Repeat exercise 3.8.8 for the hypocycloid $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos^3 t \\ a\sin^3 t \end{pmatrix}$.

3.8.10 a. Let $f : [a, b] \to \mathbb{R}$ be a smooth function satisfying $f(x) > 0$, and consider the surface obtained by rotating its graph around the x-axis. Show that the Gaussian curvature K and the mean curvature H of this surface depend only on the x-coordinate.

b. Show that $K(x) = \dfrac{-f''(x)}{f(x)\Big(1 + \big(f'(x)\big)^2\Big)^2}$.

c. Find a formula for the mean curvature in terms of f and its derivatives.

Exercise 3.8.11: The curve

$$F : t \mapsto \left[\vec{\mathbf{t}}(t), \vec{\mathbf{n}}(t), \vec{\mathbf{b}}(t)\right] = T(t)$$

is a mapping $I \mapsto SO(3)$, the space of orthogonal 3×3 matrices with determinant $+1$. So

$$t \mapsto T^{-1}(t_0)T(t)$$

is a curve in $SO(3)$ that passes through the identity at t_0.

***3.8.11** Use exercise 3.1.24 to explain why the Frenet formulas give an antisymmetric matrix.

3.8.12 Prove proposition 3.8.6, using proposition 3.8.16.

3.9 Review exercises for chapter 3

3.1 a. Show that the set $X \subset \mathbb{R}^3$ of equation

$$x^3 + xy^2 + yz^2 + z^3 = 4 \quad \text{is a smooth surface.}$$

b. Give the equations of the tangent plane and tangent space to X at $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Exercise 3.2: We strongly advocate using MATLAB or similar software.

3.2 a. For what values of c is the set of equation $Y_c = x^2 + y^3 + z^4 = c$ a smooth surface?

b. Sketch this surface for a representative sample of values of c (for instance, $-2, -1, 0, 1, 2$).

c. Give the equations of the tangent plane and tangent space at a point $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ of the surface Y_c.

3.3 Consider the space X of positions of a rod of length 2 in $\mathbb{R}^3$, where one endpoint is constrained to be on the sphere of equation $(x-1)^2+y^2+z^2=1$, and the other on the sphere of equation $(x+1)^2+y^2+z^2=1$.

a. Give equations for X as a subset of $\mathbb{R}^6$, where the coordinates in $\mathbb{R}^6$ are the coordinates $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ of the end of the rod on the first sphere, and the three coordinates $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ of the other end of the rod.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

Point for exercise 3.3, parts b and c.

b. Show that near the point in $\mathbb{R}^6$ shown in the margin, the set X is a manifold. What is the dimension of X near this point?

c. Give the equation of the tangent space to the set X, at the same point as in part b.

3.4 Consider the space X of triples $\mathbf{p} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$ such that $y \neq 0$ and the segments $\overline{\mathbf{p},\mathbf{q}}$ and $\overline{\mathbf{q},\mathbf{r}}$ form an angle of $\pi/4$.

The notation $\overline{\mathbf{p},\mathbf{q}}$ means the segment going from $\mathbf{p}$ to $\mathbf{q}$.

a. Write an equation $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ which all points of X will satisfy.

b. Show that the position of $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ where $x=0, y=1, z=1$ satisfies the equation, and that X is a smooth surface near that point.

c. True or false? Near this point, X is locally the graph of a function expressing z as a function of x and y.

d. What is the tangent plane to X at the point $x=0, y=1, z=1$? What is the tangent space to the surface at that point?

3.5 Find the Taylor polynomial of degree 3 of the function

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sin(x+y+z) \quad \text{at the point} \quad \begin{pmatrix} \pi/6 \\ \pi/4 \\ \pi/3 \end{pmatrix}.$$

3.6 Show that if $f\begin{pmatrix} x \\ y \end{pmatrix} = \varphi(x-y)$ for some twice continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$, then $D_1^2 f - D_2^2 f = 0$.

3.7 Write, to degree 3, the Taylor polynomial $P^3_{f,0}$ of

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \cos\Big(1+\sin(x^2+y)\Big) \quad \text{at the origin.}$$

***3.8** a. Show that the mapping $\varphi_1 : (\mathbb{R}^m - \{0\}) \times \mathbb{R}^{n-1}$ shown in the margin is a parametrization of the subset $U_1 \subset M_1(m,n)$ of those matrices whose first column is not $\mathbf{0}$.

$$\varphi_1\left(\mathbf{a}, \begin{bmatrix} \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}\right) \mapsto [\mathbf{a}, \lambda_2\mathbf{a}, \dots, \lambda_n\mathbf{a}]$$

Mapping for exercise 3.8, part a

b. Show that $M_1(m,n) - U_1$ is a manifold embedded in $M_1(m,n)$. What is its dimension?

c. How many parametrizations like φ_1 do you need to cover every point of $M_1(m,n)$?

***3.9** A homogeneous polynomial in two variables of degree four is an expression of the form

A homogeneous polynomial is a polynomial in which all terms have the same degree.

$$p(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4.$$

Consider the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \dfrac{p(x,y)}{x^2+y^2} & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{cases}$$

where p is a homogeneous polynomial of degree 4. What condition must the coefficients of p satisfy in order for the crossed partials $D_1(D_2(f))$ and $D_2(D_1(f))$ to be equal at the origin?

Exercise 3.11 is relevant to example 3.8.12. Hint for part b: The x-axis is contained in the surface.

3.10 a. Show that $ye^y = x$ implicitly defines y as a function of x, for $x \geq 0$.

b. Find a Taylor polynomial of the implicit function to degree 4.

3.11 a. Show that the equation $y \cos z = x \sin z$ expresses z implicitly as a function $z = g_r\begin{pmatrix} x \\ y \end{pmatrix}$ near the point $\begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$.

b. Show that $D_1 g_r \begin{pmatrix} r \\ 0 \end{pmatrix} = D_1^2 g_r \begin{pmatrix} r \\ 0 \end{pmatrix} = 0$.

$$Q_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \det \begin{bmatrix} 1 & x & y \\ 1 & y & z \\ 1 & z & x \end{bmatrix}$$

$$Q_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \det \begin{bmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 0 \end{bmatrix}$$

Functions for exercise 3.13

3.12 On $\mathbb{R}^4$ as described by $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, consider the quadratic form $Q(M) = \det M$. What is its signature?

3.13 a. Are the functions Q_1 and Q_2 in the margin quadratic forms on $\mathbb{R}^3$?

b. For any that is a quadratic form, what is its signature? Is it degenerate or nondegenerate?

3.14 Let P_k be the space of polynomials of degree at most k.

a. Show that the function $\delta_a : P_k \to \mathbb{R}$ given by $\delta_a(p) = p(a)$ is a linear function.

b. Show that $\delta_0, \dots, \delta_k$ are linearly independent. First say what it means, being careful with the quantifiers. It may help to think of the polynomial

$$x(x-1)\dots\Big(x-(j-1)\Big)\Big(x-(j+1)\Big)\dots(x-k),$$

which vanishes at $0, 1, \dots, j-1, j+1, \dots, k$ but not at j.

c. Show that the function

$$Q(p) = \Big(p(0)\Big)^2 - \Big(p(1)\Big)^2 + \dots + (-1)^k\Big(p(k)\Big)^2$$

is a quadratic form on P_k. When $k = 3$, write it in terms of the coefficients of $p(x) = ax^3 + bx^2 + cx + d$.

Exercise 3.14 part d: There is the clever way, and then there is the plodding way.

d. What is the signature of Q when $k = 3$?

3.15 Show that a 2×2 symmetric matrix $G = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ represents a positive definite quadratic form if and only if $\det G > 0$, $a + d > 0$.

3.16 Let Q be a quadratic form. Construct a symmetric matrix A as follows: each entry $A_{i,i}$ on the diagonal is the coefficient of x_i^2, while each entry $A_{i,j}$ is half the coefficient of the term x_ix_j.

Exercise 3.16, part b: Consider $Q(\vec{\mathbf{e}}_i)$ and $Q(\vec{\mathbf{e}}_i + \vec{\mathbf{e}}_j)$.

a. Show that $Q(\vec{\mathbf{x}}) = \vec{\mathbf{x}} \cdot A\vec{\mathbf{x}}$.

b. Show that A is the unique symmetric matrix with this property.

3.17 a. Find the critical points of the function $f\begin{pmatrix} x \\ y \end{pmatrix} = 3x^2 - 6xy + 2y^3$.

b. What kind of critical points are these?

Exercise 3.18, part a: This is easier if you use

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

3.18 a. What is the Taylor polynomial of degree 2 of the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \sin(2x+y) \quad \text{at the point } \begin{pmatrix} \pi/6 \\ \pi/3 \end{pmatrix}?$$

b. Show that $f\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\left(2x + y - \frac{2\pi}{3}\right) - \left(x - \frac{\pi}{6}\right)^2$ has a critical point at $\begin{pmatrix} \pi/6 \\ \pi/3 \end{pmatrix}$. What kind of critical point is it?

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \det\begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{bmatrix}$$

Function of exercise 3.19

3.19 The function in the margin has exactly five critical points.

a. Find them.

b. For each critical point, what are the quadratic terms of the Taylor polynomial at that point?

c. Say everything you can about the type of critical point each is.

3.20 a. Find the critical points of xyz, if x, y, z belong to the surface S of equation $x + y + z^2 = 16$.

b. Is there a maximum on the whole surface; if so, which critical point is it?

c. Is there a maximum on the part of S where x, y, z are all positive?

3.21 Let A, B, C, D be a convex quadrilateral in the plane, with the vertices free to move but with a the length of AB, b the length of BC, c the length of CD. and d the length of DA all assigned. Let φ be the angle at A and ψ the angle at C.

Exercise 3.21, part c: You may use the fact that a quadrilateral can be inscribed in a circle if the opposite angles add to π.

a. Show that the angles φ and ψ satisfy the constraint

$$a^2 + d^2 - 2ad\cos\varphi = b^2 + c^2 - 2bc\cos\psi.$$

b. Find a formula for the area of the quadrilateral in terms of φ, ψ and a, b, c, d.

c. Show that the area is maximum if the quadrilateral can be inscribed in a circle.

3.22 Find the maximum of the function $x_1 x_2 \dots x_n$, subject to the constraint

$$x_1^2 + 2x_2^2 + \cdots + nx_n^2 = 1.$$

3.23 Compute the Gaussian and mean curvature of the surface of equation $z = \sqrt{x^2+y^2}$ at $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ \sqrt{a^2+b^2} \end{pmatrix}$. Explain your result.

3.24 Suppose $\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{pmatrix}$ is twice continuously differentiable on a neighborhood of $[a, b]$.

a. Use Taylor's theorem with remainder (or argue directly from the mean value theorem) to show that for any $s_1 < s_2$ in $[a, b]$,

$$|\gamma(s_2) - \gamma(s_1) - \gamma'(s_1)(s_2 - s_1)| \leq C|s_2 - s_1|^2,$$

where

$$C = \sqrt{n} \sup_{j=1,\dots,n} \sup_{t\in[a,b]} |\gamma_j''(t)|.$$

b. Use this to show that

$$\lim \sum_{i=0}^{m-1} |\gamma(t_{i+1} - \gamma(t_i)| = \int_a^b |\gamma'(t)|\, dt,$$

where $a = t_0 < t_1 \cdots < t_m = b$, and we take the limit as the distances $t_{i+1} - t_i$ tend to 0.

3.25 Show that the quadratic form $ax^2 + 2bxy + cy^2$

a. is positive definite if and only if $ac - b^2 > 0$ and $a > 0$

b. is negative definite if and only if $ac - b^2 > 0$ and $a < 0$

c. has signature (1,1) if and only if $ac - b^2 < 0$.