3

Manifolds, Taylor polynomials, quadratic forms, and curvature

Thomson [Lord Kelvin] had predicted the problems of the first [transatlantic] cable by mathematics. On the basis of the same mathematics he now promised the company a rate of eight or even 12 words a minute. Half a million pounds was being staked on the correctness of a partial differential equation.—T. W. Körner, *Fourier Analysis*

3.0 Introduction

This chapter is something of a grab bag. The various themes are related, but the relationship is not immediately apparent. We begin with two sections on geometry. In section 3.1 we use the implicit function theorem to define smooth curves, smooth surfaces, and more general k-dimensional "surfaces" in $\mathbb{R}^n$, called *manifolds*. In section 3.2 we discuss linear approximations to manifolds: *tangent spaces*.

When a computer calculates sines, it is not looking up the answer in some mammoth table of sines; stored in the computer is a polynomial that very well approximates $\sin x$ for x in some particular range. Specifically, it uses the formula

$$\sin x = x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + a_{11}x^{11} + \epsilon(x),$$

where the coefficients are

$$\begin{aligned} a_3 &= -.1666666664 \\ a_5 &= .0083333315 \\ a_7 &= -.0001984090 \\ a_9 &= .0000027526 \\ a_{11} &= -.0000000239. \end{aligned}$$

When $|x| \le \pi/2$, the error is guaranteed to be less than 2×10^{-9}, good enough for a calculator that computes to eight significant digits.

We switch gears in section 3.3, where we use higher partial derivatives to construct the Taylor polynomial of a function in several variables. We saw in section 1.7 how to approximate a nonlinear function by its derivative; here we will see that, as in one dimension, we can make higher-degree approximations using a function's Taylor polynomial. This is useful, since polynomials, unlike sines, cosines, exponentials, square roots, logarithms, ... can actually be computed using arithmetic. Computing Taylor polynomials by calculating higher partial derivatives can be quite unpleasant; in section 3.4 we give rules for computing them by combining Taylor polynomials of simpler functions.

In section 3.5 we take a brief detour, introducing quadratic forms and seeing how to classify them according to their "signature." We see in section 3.6 that if we consider the second-degree terms of a function's Taylor polynomial as a quadratic form, its signature usually tells us whether at a point where the derivative vanishes the function has a minimum value, a maximum value, or some kind of *saddle*, where the function goes up in some directions and down in others, as in a mountain pass. In section 3.7 we look at extrema of a function restricted to some manifold $M \subset \mathbb{R}^n$.

In section 3.8 we give a brief introduction to the vast and important subject of the geometry of curves and surfaces, using the higher-degree approximations provided by Taylor polynomials: the *curvature* of a curve or surface depends on the quadratic terms of the functions defining it, and the *torsion* of a space curve depends on the cubic terms.

3.1 Manifolds

Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions
—Felix Klein

These familiar objects are by no means simple: already, the theory of soap bubbles is a difficult topic, with a complicated partial differential equation controlling the shape of the film.

Figure 3.1.1.
Felix Klein (1849–1925)
Klein's work in geometry "has become so much a part of our present mathematical thinking that it is hard for us to realise the novelty of his results."—From a biographical sketch by J. O'Connor and E. F. Robertson. Klein was also instrumental in developing *Mathematische Annalen* into one of the most prestigious mathematical journals.

In this section we introduce one more actor in multivariable calculus. So far, our mappings have been first linear, then nonlinear with good linear approximations. But the domain and codomain of our mappings have been "flat" open subsets of $\mathbb{R}^n$. Now we want to allow "nonlinear $\mathbb{R}^n$'s," called *smooth manifolds*.

Manifolds are a generalization of the familiar curves and surfaces of everyday experience. A one-dimensional manifold is a smooth curve; a two-dimensional manifold is a smooth surface. Smooth curves are idealizations of things like telephone wires or a tangled garden hose. Particularly beautiful smooth surfaces are produced when you blow soap bubbles that wobble and slowly vibrate as they drift through the air. Other examples are shown in figure 3.1.2.

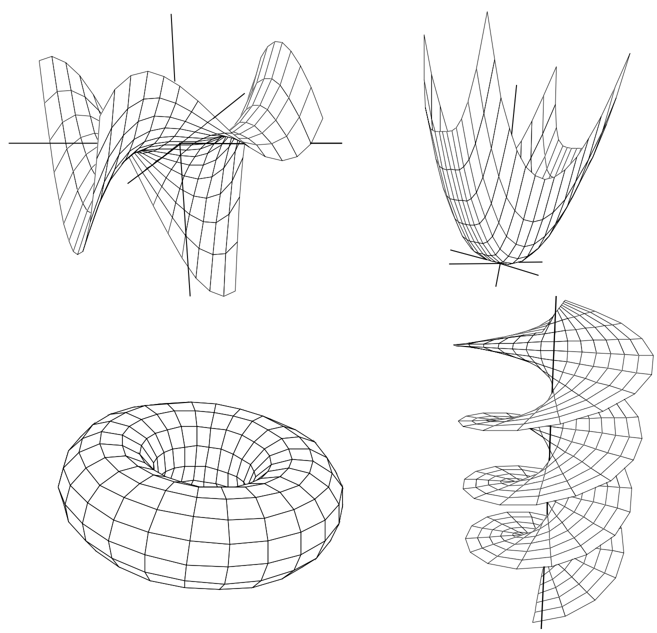

Figure 3.1.2. Four surfaces in $\mathbb{R}^3$. The top two are graphs of functions. The bottom two are *locally* graphs of functions. All four qualify as smooth surfaces (two-dimensional manifolds) under definition 3.1.2.

We will define smooth manifolds mathematically, excluding some objects that we might think of as smooth: a figure eight, for example. We will see how to use the implicit function theorem to tell whether the locus defined by an equation is a smooth manifold. Finally, we will compare knowing a manifold by equations, and knowing it by a parametrization.

Smooth manifolds in $\mathbb{R}^n$

When is a subset $X \subset \mathbb{R}^n$ a smooth manifold? Our definition is based on the notion of graph.

Remember from the discussion of set theory notation (section 0.3) that $A \times B$ is the set of pairs (a, b) with $a \in A$ and $b \in B$. Here $\mathbf{x}$ is a point in $\mathbb{R}^n$ and $\mathbf{y}$ is a point in $\mathbb{R}^m$. The graph of such a function consists of points $\begin{pmatrix} \mathbf{x} \\ \mathbf{f}(\mathbf{x}) \end{pmatrix}$ in $\mathbb{R}^{n+m}$. (This is the simplest way to think of it, with the n active variables coming first, followed by the m passive variables.

A manifold M embedded in $\mathbb{R}^n$, denoted $M \subset \mathbb{R}^n$, is sometimes called a *submanifold* of $\mathbb{R}^n$. Strictly speaking, it should not be referred to simply as a "manifold," which can mean an abstract manifold, not embedded in any space. The manifolds in this book are all submanifolds of $\mathbb{R}^n$.

With this definition, which depends on chosen coordinates, it isn't obvious that if you rotate a smooth manifold it is still smooth. We will see in theorem 3.1.16 that it is.

Definition 3.1.1 (Graph). The *graph* $\Gamma(\mathbf{f})$ of a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is the set of pairs $(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^n \times \mathbb{R}^m)$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$.

You are familiar with graphs of functions $f : \mathbb{R} \to \mathbb{R}$; most often we graph such functions with the horizontal x-axis corresponding to the input, and the vertical axis corresponding to values of f at different x. Note that the graph of such a function is a subset of $\mathbb{R}^2$. For example, the graph of $f(x) = x^2$ consists of the points $\begin{pmatrix} x \\ f(x) \end{pmatrix} \in \mathbb{R}^2$, i.e., the points $\begin{pmatrix} x \\ x^2 \end{pmatrix}$.

The top two surfaces shown in figure 3.1.2 are graphs of functions from $\mathbb{R}^2$ to $\mathbb{R}$: the surface on the left is the graph of $f\begin{pmatrix} x \\ y \end{pmatrix} = x^3 - 2xy^2$; that on the right is the graph of $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^4$. Although we depict these graphs on a flat piece of paper, they are actually subsets of $\mathbb{R}^3$. The first consists of the points $\begin{pmatrix} x \\ y \\ x^3 - 2xy^2 \end{pmatrix}$, the second of the points $\begin{pmatrix} x \\ y \\ x^2 + y^4 \end{pmatrix}$.

More generally, the graph of a function $\mathbf{f}$ lives in a space whose dimension is the sum of the dimensions of the domain and codomain of $\mathbf{f}$: the graph of a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is a subset of $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$.

Definition 3.1.2 says that if such a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is C^1, then its graph is a smooth n-dimensional manifold in $\mathbb{R}^{n+m}$. Thus the top two graphs shown in figure 3.1.2 are two-dimensional manifolds in $\mathbb{R}^3$.

But the torus and helicoid shown in figure 3.1.2 are also two-dimensional manifolds. Neither one is the graph of a single function expressing one variable in terms of the other two. But both are *locally* graphs of functions.

Definition 3.1.2 (Smooth manifold in $\mathbb{R}^n$). A subset $M \subset \mathbb{R}^n$ is a smooth k-dimensional manifold if locally it is the graph of a C^1 mapping expressing $n - k$ variables as functions of the other k variables.

Generally, "smooth" means "as many times differentiable as is relevant to the problem at hand." In this and the next section, it means "of class C^1." (Some authors use "smooth" to mean C^∞: "infinitely many times differentiable." For our purposes this is overkill.) When speaking of smooth manifolds, we often omit the word smooth.

Especially in higher dimensions, making some kind of global sense of a patchwork of graphs of functions can be quite challenging indeed; a mathematician trying to picture a manifold is rather like a blindfolded person who has never met or seen a picture of an elephant seeking to identify one by patting first an ear, then the trunk or a leg. It is a subject full of open questions, some fully as interesting and demanding as, for example, Fermat's last theorem, whose solution after more than three centuries aroused such passionate interest.

Three-dimensional and four-dimensional manifolds are of particular interest, in part because of applications in representing space-time.

"Locally" means that every point $\mathbf{x} \in M$ has a neighborhood $U \subset \mathbb{R}^n$ such that $M \cap U$ (the part of M in U) is the graph of a mapping expressing $n-k$ of the coordinates of each point in $M \cap U$ in terms of the other k. This may sound like an unwelcome complication, *but if we omitted the word "locally" then we would exclude from our definition most interesting manifolds.* We already saw that neither the torus nor the helicoid of figure 3.1.2 is the graph of a single function expressing one variable as a function of the other two. Even such a simple curve as the unit circle is not the graph of a single function expressing one variable in terms of the other. In figure 3.1.3 we show another smooth curve that would not qualify as a manifold if we required it to be the graph of a single function expressing one variable in terms of the other; the caption justifies our claim that this curve is a smooth curve.

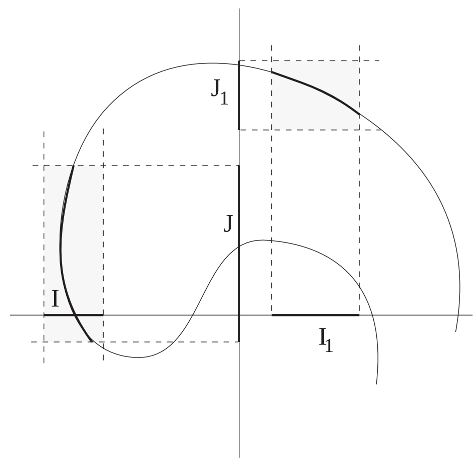

FIGURE 3.1.3. Above, I and I_1 are intervals on the x-axis; J and J_1 are intervals on the y-axis. The darkened part of the curve in the shaded rectangle $I \times J$ is the graph of a function expressing $x \in I$ as a function of $y \in J$, and the darkened part of the curve in $I_1 \times J_1$ is the graph of a function expressing $y \in J_1$ as a function of $x \in I_1$. (By decreasing the size of J_1 a bit, we could also think of the part of the curve in $I_1 \times J_1$ as the graph of a function expressing $x \in I_1$ as a function of $y \in J_1$.) But we cannot think of the darkened part of the curve in $I \times J$ as the graph of a function expressing $y \in J$ as a function of $x \in I$; there are values of x that give two different values of y, and others that give none, so such a "function" is not well defined.

Example 3.1.3 (Graph of smooth function is smooth manifold). The graph of any smooth function is a smooth manifold. The curve of equation $y = x^2$ is a one-dimensional manifold: the graph of y as the function $f(x) = x^2$. The curve of equation $x = y^2$ is also a one-dimensional manifold: the graph of a function representing x as a function of y. Each surface at the top of figure 3.1.2 is the graph of a function representing z as a function of x and y. △