2.11 Review exercises for chapter 2

2.1 a. For what values of a and b does the system of linear equations

$$\begin{aligned} x + y - z &= a \\ x + 2z &= b \\ x + ay + z &= b \end{aligned}$$

have one solution? No solutions? Infinitely many solutions?

b. For what values of a and b is the matrix of coefficients invertible?

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 3 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

Matrix A of exercise 2.2

2.2 When A is the matrix at left, multiplication by what elementary matrix corresponds to

a. Exchanging the first and second rows of A?

b. Multiplying the fourth row of A by 3?

c. Adding 2 times the third row of A to the first row of A?

2.3 a. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Are the following statements true or false?

1. If $\ker T = 0$, and $T(\vec{\mathbf{y}}) = \vec{\mathbf{b}}$, then $\vec{\mathbf{y}}$ is the only solution to $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$.
2. If $\vec{\mathbf{y}}$ is the only solution to $T(\vec{\mathbf{x}}) = \vec{\mathbf{c}}$, then for any $\vec{\mathbf{b}} \in \mathbb{R}^m$, a solution exists to $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$.
3. If $\vec{\mathbf{y}} \in \mathbb{R}^n$ is a solution to $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$, it is the only solution.
4. If for any $\vec{\mathbf{b}} \in \mathbb{R}^m$ the equation $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$ has a solution, then it is the only solution.

b. For any statements that are false, can one impose conditions on m and n that make them true?

2.4 a. Row reduce the matrix A in the margin.

$$\begin{bmatrix} 1 & -1 & 3 & 0 & -2 \\ -2 & 2 & -6 & 0 & 4 \\ 0 & 2 & 5 & -1 & 0 \\ 2 & -6 & -4 & 2 & -4 \end{bmatrix}$$

Matrix A for exercise 2.4.

b. Let $\vec{\mathbf{v}}_m,\ m = 1, \dots, 5$ be the columns of A. What can you say about the systems of equations

$$[\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k] \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \vec{\mathbf{v}}_{k+1} \quad \text{for } k = 1, 2, 3, 4?$$

Exercise 2.4, part b: For example, for $k = 2$ we are asking about the system of equations

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 0 & 2 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 5 \\ -4 \end{bmatrix}.$$

2.5 a. Show that if A is an invertible $n \times n$ matrix, B is an invertible $m \times m$ matrix, and C is any $n \times m$ matrix, then the $(n+m) \times (n+m)$ matrix

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}, \quad \text{where 0 stands for the } m \times n \text{ 0 matrix, is invertible.}$$

b. Find a formula for the inverse.

2.6 In exercise 2.2.11 you were asked to show that using row reduction to solve n equations in n unknowns takes $n^3 + n^2/2 - n/2$ operations, where a single addition, multiplication, or division counts as one operation. How many operations does it take to compute the inverse of an $n \times n$ matrix A? How many operations does it take to perform the matrix multiplication $A^{-1}\vec{\mathbf{b}}$?

2.7 a. For what values of a is the matrix $\begin{bmatrix} 1 & -1 & -1 \\ 0 & a & 1 \\ 2 & a+2 & a+2 \end{bmatrix}$ invertible?

b. For those values, compute the inverse.

2.8 Show that the following two statements are equivalent to saying that a set of vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ is linearly independent:

a. The only way to write the zero vector $\vec{\mathbf{0}}$ as a linear combination of the $\vec{\mathbf{v}}_i$ is to use only zero coefficients.

b. None of the $\vec{\mathbf{v}}_i$ is a linear combination of the others.

2.9 a. Show that $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$ form an orthonormal basis of $\mathbb{R}^2$.

b. Show that $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \begin{bmatrix}\sin\theta\\ -\cos\theta\end{bmatrix}$ form an orthonormal basis of $\mathbb{R}^2$.

c. Show that any orthogonal 2×2 matrix gives either a reflection or a rotation: a reflection if its determinant is negative, a rotation if its determinant is positive.

2.10 a. For vectors in $\mathbb{R}^2$, prove that the length squared of a vector is the sum of the squares of its coordinates, with respect to any orthonormal basis.

b. Prove the same thing for vectors in $\mathbb{R}^3$.

c. Repeat for $\mathbb{R}^n$: show that if $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$ and $\vec{\mathbf{w}}_1, \dots, \vec{\mathbf{w}}_n$ are two orthonormal bases, and $a_1\vec{\mathbf{v}}_1 + \dots + a_n\vec{\mathbf{v}}_n = b_1\vec{\mathbf{w}}_1 + \dots + b_n\vec{\mathbf{w}}_n$, then

$$a_1^2 + \dots + a_n^2 = b_1^2 + \dots + b_n^2.$$

2.11 a. Let $A = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$. Are the elements I, A, A^2, A^3 linearly independent in $\operatorname{Mat}(2,2)$? What is the dimension of the subspace $V \subset \operatorname{Mat}(2,2)$ that they span? (Recall that $\operatorname{Mat}(n,m)$ denotes the set of $n \times m$ matrices.)

b. Show that the set W of matrices $B \in \operatorname{Mat}(2,2)$ that satisfy $AB = BA$ is a subspace of $\operatorname{Mat}(2,2)$. What is its dimension?

c. Show that $V \subset W$. Are they equal?

2.12 Let $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ be vectors in $\mathbb{R}^n$, and set $V = [\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]$.

a. Show that the set $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ is orthogonal if and only if $V^\top V$ is diagonal.

b. Show that the set $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ is orthonormal if and only if $V^\top V = I_k$.

2.13 Find a basis for the image and the kernel of the matrices

$$A = \begin{bmatrix}1 & 1 & 3 & 6 & 2\\ 2 & -1 & 0 & 4 & 1\\ 4 & 1 & 6 & 16 & 5\end{bmatrix} \qquad B = \begin{bmatrix}2 & 1 & 3 & 6 & 2\\ 2 & -1 & 0 & 4 & 1\end{bmatrix},$$

and verify that the dimension formula is true.

Exercise 2.14: For example, the polynomial

$$p = 2x - y + 3xy + 5y^2$$

corresponds to the point $\begin{pmatrix}0\\2\\-1\\0\\3\\5\end{pmatrix}$, so

$$xD_1p = x(2+3y) = 2x + 3xy$$

corresponds to $\begin{pmatrix}0\\2\\0\\0\\3\\0\end{pmatrix}$.

2.14 Let P be the space of polynomials of degree at most 2 in the two variables x, y, which we will identify to $\mathbb{R}^6$ by identifying

$$a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \quad \text{with} \quad \begin{pmatrix}a_1\\ \vdots\\ a_6\end{pmatrix}.$$

a. What are the matrices of the linear transformations $S, T : P \to P$

$$S(p)\begin{pmatrix}x\\y\end{pmatrix} = xD_1p\begin{pmatrix}x\\y\end{pmatrix} \quad \text{and} \quad T(p)\begin{pmatrix}x\\y\end{pmatrix} = yD_2p\begin{pmatrix}x\\y\end{pmatrix}?$$

b. What are the kernel and the image of the linear transformation

$$p \mapsto 2p - S(p) - T(p)?$$

2.15 Let $a_1, \dots, a_k, b_1, \dots, b_k$ be any $2k$ numbers. Show that there exists a unique polynomial p of degree at most $2k-1$ such that

$$p(i) = a_i, \quad p'(i) = b_i$$

for all integers i with $1 \le i \le k$. In other words, show that the values of p and p' at $1, \dots, k$ determine p. *Hint*: You should use the fact that a polynomial p of degree d such that $p(i) = p'(i) = 0$ can be written $p(x) = (x-i)^2 q(x)$ for some polynomial q of degree $d-2$.

2.16 A square $n \times n$ matrix P such that $P^2 = P$ is called a *projection*.

a. Show that P is a projection if and only if $I - P$ is a projection. Show that if P is invertible, then P is the identity.

Hint for exercise 2.16, part b: $\vec{\mathbf{v}} = P\vec{\mathbf{v}} + (\vec{\mathbf{v}} - P\vec{\mathbf{v}})$.

b. Let $V_1 = \operatorname{img} P$ and $V_2 = \ker P$. Show that any vector $\vec{\mathbf{v}} \in \mathbb{R}^n$ can be written uniquely $\vec{\mathbf{v}} = \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2$ with $\vec{\mathbf{v}}_1 \in V_1$ and $\vec{\mathbf{v}}_2 \in V_2$.

c. Show that there exist a basis $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$ of $\mathbb{R}^n$ and a number $k \le n$ such that

$$\begin{aligned} P\vec{\mathbf{v}}_1 &= \vec{\mathbf{v}}_1 \\ P\vec{\mathbf{v}}_2 &= \vec{\mathbf{v}}_2 \\ &\vdots \\ P\vec{\mathbf{v}}_k &= \vec{\mathbf{v}}_k \end{aligned} \qquad \text{and} \qquad \begin{aligned} P\vec{\mathbf{v}}_{k+1} &= \mathbf{0} \\ P\vec{\mathbf{v}}_{k+2} &= \mathbf{0} \\ &\vdots \\ P\vec{\mathbf{v}}_n &= \mathbf{0}. \end{aligned}$$

*d. Show that if P_1 and P_2 are projections such that $P_1P_2 = 0$, then $Q = P_1 + P_2 - (P_2P_1)$ is a projection, $\ker Q = \ker P_1 \cap \ker P_2$, and the image of Q is the space spanned by the image of P_1 and the image of P_2.

Exercise 2.17: Recall that $\mathcal{C}^2$ is the space of C^2 (twice continuously differentiable) functions.

2.17 Show that the transformation $T : \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ given by formula 2.6.8 in example 2.6.8 is a linear transformation.

2.18 Denote by $\mathcal{L}\Big(\operatorname{Mat}(n,n), \operatorname{Mat}(n,n)\Big)$ the space of linear transformations from $\operatorname{Mat}(n,n)$ to $\operatorname{Mat}(n,n)$.

a. Show that $\mathcal{L}\Big(\operatorname{Mat}(n,n), \operatorname{Mat}(n,n)\Big)$ is a vector space and that it is finite dimensional. What is its dimension?

b. Prove that for any $A \in \operatorname{Mat}(n,n)$, the transformations

$$L_A,\ R_A : \operatorname{Mat}(n,n) \to \operatorname{Mat}(n,n)$$

given by $L_A(B) = AB, \quad R_A(B) = BA$ are linear transformations.

c. Let $\mathcal{M}_L \subset \mathcal{L}\Big(\operatorname{Mat}(n,n), \operatorname{Mat}(n,n)\Big)$ be the set of functions of the form L_A. Show that it is a subspace of $\mathcal{L}\Big(\operatorname{Mat}(n,n), \operatorname{Mat}(n,n)\Big)$. What is its dimension?

d. Show that there are linear transformations $T : \operatorname{Mat}(2,2) \to \operatorname{Mat}(2,2)$ that cannot be written as $L_A + R_B$. Can you find an explicit one?

e. What are $|L_A|$ and $|R_A|$ in terms of $|A|$ and $|B|$?

2.19 Show that in a vector space of dimension n, more than n vectors are never linearly independent, and fewer than n vectors never span.

2.20 Suppose we use the same operator $T : P_2 \to P_2$ as in exercise 2.6.8, but choose instead to work with the basis

$$q_1(x) = x^2, \quad q_2(x) = x^2 + x, \quad q_3(x) = x^2 + x + 1.$$

Now what is the matrix $\Phi_{\{q\}}^{-1} \circ T \circ \Phi_{\{q\}}$?

2.21 Let $V, W \subset \mathbb{R}^n$ be two subspaces.

a. Show that $V \cap W$ is a subspace of $\mathbb{R}^n$.

b. Show that if $V \cup W$ is a subspace of $\mathbb{R}^n$, then either $V \subset W$ or $W \subset V$.

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(x-y)+y^2 \\ \cos(x+y)-x \end{pmatrix}$$

Map $F : \mathbb{R}^2 \to \mathbb{R}^2$ for exercise 2.22

2.22 a. Find a global Lipschitz ratio for the derivative of the map F defined in the margin.

b. Do one step of Newton's method to solve

$$F\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} .5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{starting at } \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

c. Can you be sure that Newton's method converges?

Exercise 2.23: Note that

$$[2\,I]^3 = [8\,I], \quad \text{i.e.,}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

2.23 Using Newton's method, solve the equation $A^3 = \begin{bmatrix} 9 & 0 & 1 \\ 0 & 7 & 0 \\ 0 & 2 & 8 \end{bmatrix}$.

Exercise 2.24: The computation really does require you to row reduce a 4×4 matrix.

2.24 Consider the map $F : \text{Mat}\,(2,2) \to \text{Mat}\,(2,2)$ given by $F(A) = A^2 + A^{-1}$. Set $A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B_0 = F(A_0)$, and define

$$U_r = \{\, B \in \text{Mat}\,(2,2) \mid |B - B_0| < r \,\}.$$

Does there exist $r > 0$ and a differentiable mapping $G : U_r \to \text{Mat}\,(2,2)$ such that $F(G(B)) = B$ for every $B \in U_r$?

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y - 2 \\ y^2 - x - 6 \end{pmatrix}$$

Mapping F for exercise 2.25

2.25 a. Find a global Lipschitz constant for the derivative of the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given in the margin.

b. Do one step of Newton's method to solve $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ starting at $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

c. Find and sketch a disc in $\mathbb{R}^2$ which you are sure contains a root.

2.26 There are other plausible ways to measure matrices other than the length and the norm; for example, we could declare the size $|A|$ of a matrix A to be the largest absolute value of an entry. In this case, $|A+B| \le |A| + |B|$, but the statement $|A\vec{\mathbf{x}}| \le |A||\vec{\mathbf{x}}|$ (where $|\vec{\mathbf{x}}|$ is the ordinary length of a vector) is false. Find an ϵ so that it is false for

$$A = \begin{bmatrix} 1 & 1 & 1+\epsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The norm $\|A\|$ of a matrix A is defined in section 2.9 (definition 2.9.6).

2.27 Show that $\|A\| = \|A^\top\|$.

2.28 In example 2.10.8 we found that $M = 2\sqrt{2}$ is a global Lipschitz constant for the function $\mathbf{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(x+y) \\ x^2 - y^2 \end{pmatrix}$. What Lipschitz constant do you get using the method of second partial derivatives? Using that Lipschitz constant, what minimum domain do you get for the inverse function at $\begin{pmatrix} 0 \\ \pi \end{pmatrix}$?

2.29 a. True or false? The equation $\sin(xyz) = z$ expresses x implicitly as a differentiable function of y and z near the point $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \pi/2 \\ 1 \\ 1 \end{pmatrix}$.

b. True or false? The equation $\sin(xyz) = z$ expresses z implicitly as a differentiable function of x and y near the same point.

Exercise 2.30: You may use the fact that if
$$S : \text{Mat}\,(2,2) \to \text{Mat}\,(2,2)$$
is the squaring map
$$S(A) = A^2,$$
then
$$[\mathbf{D}S(A)]B = AB + BA.$$

Exercise 2.32, part a: It may be easier to work over the complex numbers.

Relatively prime: with no common factors.

2.30 True or false? There exists a neighborhood $U \subset \text{Mat}\,(2,2)$ of $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ and a C^1 mapping $F : U \to \text{Mat}\,(2,2)$ with
$$F\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad \text{and} \quad (F(A))^2 = A.$$

2.31 True or false? (Explain your answer.) There exists $r > 0$ and a differentiable map $g : B_r\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right) \to \text{Mat}\,(2,2)$ such that
$$g\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$$
and $(g(A))^2 = A$for all $A \in B_r\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right)$.

***2.32** This exercise gives a proof of *Bezout's theorem*. Let p_1 and p_2 be polynomials of degree k_1 and k_2 respectively, and consider the mapping
$$T\colon (q_1, q_2) \to p_1 q_1 + p_2 q_2,$$
where q_1 and q_2 are polynomials of degree at most $k_2 - 1$ and $k_1 - 1$ respectively, so that $p_1q_1 + p_2q_2$ is of degree $\le k_1 + k_2 - 1$.

Note that the space of such (q_1, q_2) is of dimension $k_1 + k_2$, and the space of polynomials of degree at most $k_1 + k_2 - 1$ is also of dimension $k_1 + k_2$.

a. Show that $\ker T = \{0\}$ if and only if p_1 and p_2 are *relatively prime*.

b. Use corollary 2.5.10 to prove *Bezout's identity*: if p_1, p_2 are relatively prime, then there exist unique q_1 and q_2 of degree at most $k_2 - 1$ and $k_1 - 1$ such that $p_1q_1 + p_2q_2 = 1$.