A.15 Stirling's formula and the central limit theorem

Stirling's formula was actually due to the English mathematician Abraham de Moivre (1667–1754), not the Scottish mathematician James Stirling (1692–1770), but de Moivre credited Stirling with improving it.

To see why the central limit theorem is true, we need to understand how the factorial $n!$ behaves as n becomes large. How big is $100!$? How many digits does it have? *Stirling's formula* gives a very useful approximation.

Proposition A15.1 (Stirling's formula). *The number $n!$ is approximately*

$$n! \approx \sqrt{2\pi}\left(\frac{n}{e}\right)^n \sqrt{n}, \qquad \text{A15.1}$$

in the sense that the ratio of the two sides tends to 1 as n tends to ∞.

For instance,

$$\sqrt{2\pi}\,(100/e)^{100}\sqrt{100} \approx 9.3248 \cdot 10^{157} \text{ and } 100! \approx 9.3326 \cdot 10^{157}, \qquad \text{A15.2}$$

for a ratio of about 1.0008.

Proof. Define the number R_n by the formula

$$\ln n! = \underbrace{\ln 1 + \ln 2 + \cdots + \ln n}_{\text{midpoint Riemann sum}} = \int_{1/2}^{n+1/2} \ln x\, dx + R_n. \qquad \text{A15.3}$$

(As illustrated by figure A15.1, the left side is a midpoint Riemann sum.) This formula is justified by the following computation, which shows that the R_n form a convergent sequence:

$$|R_n - R_{n-1}| = \left|\ln n - \int_{n-1/2}^{n+1/2} \ln x\, dx\right| = \left|\int_{-1/2}^{1/2} \ln\left(1+\frac{t}{n}\right) dt\right| \qquad \text{A15.4}$$

$$= \left|\int_{-1/2}^{1/2}\left(\ln\left(1+\frac{t}{n}\right) - \frac{t}{n}\right) dt\right| \le \left|-2\int_{-1/2}^{1/2}\left(\frac{t}{n}\right)^2 dt\right| = \frac{1}{6n^2},$$

Equation A15.4: The second equality comes from setting $x = n+t$ and writing

$$x = n + t = n\left(1+\frac{t}{n}\right),$$

so

$$\ln x = \ln\Big(n(1+t/n)\Big) = \ln n + \ln(1+t/n)$$

and

$$\int_{-1/2}^{1/2} \ln n\, dt = \ln n.$$

The next is justified by

$$\int_{-1/2}^{1/2} \frac{t}{n} dt = 0.$$

The inequality is Taylor's theorem with remainder:

$$\ln(1+h) = h + \frac{1}{2}\left(-\frac{1}{(1+c)^2}\right)h^2$$

for some c with $|c| < |h|$; in our case, $h = t/n$ with $t \in [-1/2, 1/2]$ and $c = -1/2$ is the worst value. (See theorem A12.2, applied to $f(a+h) = \ln(1+\frac{t}{n})$ and $k = 1$.)

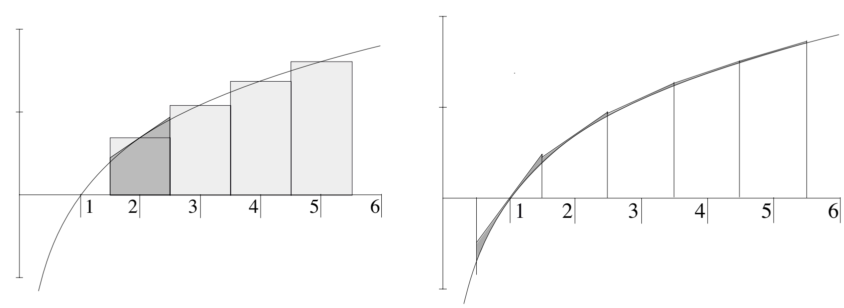

Figure A15.1. Left: The sum $\ln 1 + \ln 2 + \cdots + \ln n$ is a midpoint Riemann sum for the integral $\int_{1/2}^{n+1/2} \ln x\, dx$. The kth rectangle has the same area as the trapezoid whose top edge is tangent to the graph of $\ln x$ at $\ln n$, as illustrated when $k = 2$. Right: The difference between the areas of the trapezoids and the area under the graph of the logarithm is the shaded region. It has finite total area, as shown in equation A15.3.

so the series formed by the $R_n - R_{n-1}$ is convergent, and the sequence converges to some limit R. Thus we can rewrite equation A15.3 as follows:

$$\begin{aligned}\ln n! &= \int_{1/2}^{n+1/2} \ln x\,dx + R + \epsilon_1(n) \\ &= \big[x \ln x - x\big]_{1/2}^{n+1/2} + R + \epsilon_1(n) \qquad \text{A15.5} \\ &= \left(\left(n+\frac{1}{2}\right)\ln\left(n+\frac{1}{2}\right) - \left(n+\frac{1}{2}\right)\right) - \left(\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\right) + R + \epsilon_1(n),\end{aligned}$$

where $\epsilon_1(n)$ tends to 0 as n tends to ∞. Now notice that

$$\left(n+\frac{1}{2}\right)\ln\left(n+\frac{1}{2}\right) = \left(n+\frac{1}{2}\right)\ln n + \frac{1}{2} + \epsilon_2(n), \qquad \text{A15.6}$$

where $\epsilon_2(n)$ includes all the terms that tend to 0 as $n \to \infty$. Putting all this together, we see that there is a constant

$$c = R - \left(\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\right) \quad \text{such that} \qquad \text{A15.7}$$

$$\ln n! = n \ln n + \frac{1}{2}\ln n - n + c + \overbrace{\epsilon(n)}^{\epsilon_1(n)+\epsilon_2(n)}, \qquad \text{A15.8}$$

where $\epsilon(n) \to 0$ as $n \to \infty$. Exponentiating this gives exactly Stirling's formula, except for the determination of the constant C:

$$n! = C n^n e^{-n}\sqrt{n} \underbrace{e^{\epsilon(n)}}_{\to 1 \text{ as } n\to\infty}, \qquad \text{A15.9}$$

where $C = e^c$. There isn't any obvious reason why it should be possible to evaluate C exactly, but it turns out that $C = \sqrt{2\pi}$, as we will see in equation A15.21. □

Equation A15.6 comes from:

$$\begin{aligned}\ln\left(n+\frac{1}{2}\right) &= \ln\left(n\left(1+\frac{1}{2n}\right)\right) \\ &= \ln n + \ln\left(1+\frac{1}{2n}\right) \\ &= \ln n + \frac{1}{2n} + O\left(\frac{1}{n^2}\right).\end{aligned}$$

The epsilons $\epsilon_1(n)$ and $\epsilon_2(n)$ are unrelated, but both go to 0 as $n \to \infty$, as does

$$\epsilon(n) = \epsilon_1(n) + \epsilon_2(n).$$

When James Stirling was 17, his father was arrested for high treason (but later acquitted) because of Jacobite sympathies; later Stirling lost his scholarship at Oxford University because he refused to take an oath of allegiance to the English throne.

He spent several years in Italy, where, according to one account, he learned the secrets of Venetian glassmakers, who tried to murder him to prevent their secrets from being divulged.

He then spent ten years in London, where he was a friend of Newton's; Newton proposed Stirling for a fellowship of the Royal Society of London. In 1735 he became manager of a mining company and was so busy that it took him two years to respond to a letter from Euler in 1736, in which Euler wrote, "the more I have learned from your excellent articles ... concerning the nature of series, a study in which I have indeed expended much effort, the more I have wished to become acquainted with you."

We now prove the following version of the central limit theorem:

Theorem A15.2 (Central limit theorem). *If a fair coin is tossed $2n$ times, the probability that the number of heads is between $n + a\sqrt{n}$ and $n + b\sqrt{n}$ tends to*

$$\frac{1}{\sqrt{\pi}}\int_a^b e^{-t^2}\,dt \quad \text{as } n \text{ tends to } \infty. \qquad \text{A15.10}$$

Proof. The probability of having between $n + a\sqrt{n}$ and $n + b\sqrt{n}$ heads is

$$\frac{1}{2^{2n}}\sum_{k=a\sqrt{n}}^{b\sqrt{n}} \binom{2n}{n+k} = \frac{1}{2^{2n}}\sum_{k=a\sqrt{n}}^{b\sqrt{n}} \frac{(2n)!}{(n+k)!(n-k)!}. \qquad \text{A15.11}$$

The idea is to use Stirling's formula to rewrite the sum on the right, cancel everything we can, and see that what is left is a Riemann sum for the integral in equation A15.10 (more precisely, $1/\sqrt{\pi}$ times that Riemann sum).

We use the version of Stirling's formula given in equation A15.9, using C rather than $\sqrt{2\pi}$, as we have not yet proved that $C = \sqrt{2\pi}$.

Let us begin by writing $k = t\sqrt{n}$, so that the sum is over those values of t between a and b such that $t\sqrt{n}$ is an integer; we will denote this set by $T_{[a,b]}$. These points are regularly spaced, $1/\sqrt{n}$ apart, between a and b, and hence are good candidates for the points at which to evaluate a function when forming a Riemann sum. With this notation, our sum becomes

$$\frac{1}{2^{2n}} \sum_{t \in T_{[a,b]}} \frac{(2n)!}{(n+t\sqrt{n})!(n-t\sqrt{n})!} \qquad A15.12$$

$$\approx \frac{1}{2^{2n}} \sum_{t \in T_{[a,b]}} \frac{C(2n)^{2n}e^{-2n}\sqrt{2n}}{\left(C(n+t\sqrt{n})^{(n+t\sqrt{n})}e^{-(n+t\sqrt{n})}\sqrt{n+t\sqrt{n}}\right)\left(C(n-t\sqrt{n})^{(n-t\sqrt{n})}e^{-(n-t\sqrt{n})}\sqrt{n-t\sqrt{n}}\right)}.$$

Now for some of the cancellations: $(2n)^{2n} = 2^{2n}n^{2n}$, and the powers of 2 cancel with the fraction in front of the sum. Also, all the exponential terms cancel, since $e^{-(n+t\sqrt{n})}e^{-(n-t\sqrt{n})} = e^{-2n}$. Also, one power of C cancels. This leaves

$$\cdots = \frac{1}{C} \sum_{t \in T_{[a,b]}} \frac{n^{2n}\sqrt{2n}}{\sqrt{n^2-t^2n}\ (n+t\sqrt{n})^{(n+t\sqrt{n})}(n-t\sqrt{n})^{(n-t\sqrt{n})}}. \qquad A15.13$$

Next, write $(n+t\sqrt{n})^{(n+t\sqrt{n})} = n^{(n+t\sqrt{n})}(1+t/\sqrt{n})^{(n+t\sqrt{n})}$, and similarly for the term in $n - t\sqrt{n}$, note that the powers of n cancel with the n^{2n} in the numerator, to find

$$\cdots = \frac{1}{C} \sum_{t \in T_{[a,b]}} \underbrace{\sqrt{\frac{2n}{n^2-t^2n}}}_{\text{base}} \underbrace{\frac{1}{(1+t/\sqrt{n})^{(n+t\sqrt{n})}(1-t/\sqrt{n})^{(n-t\sqrt{n})}}}_{\text{height of rectangles for Riemann sum}}. \qquad A15.14$$

We denote by Δt the spacing of the points t (i.e., $1/\sqrt{n}$).

As $n \to \infty$, the term under the square root converges to $\sqrt{2/n} = \sqrt{2}\Delta t$, so it is the length of the base of the rectangles we need for our Riemann sum. For the other, remember that

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a. \qquad A15.15$$

We use equation A15.15 repeatedly in the following calculation:

Third line of equation A15.16: The denominator of the first term tends to e^{-t^2} as $n \to \infty$, by equation A15.15. By the same equation, the numerator of the second term tends to e^{-t^2}, and the denominator of the second term tends to e^{t^2}.

$$\begin{aligned} &\frac{1}{(1+t/\sqrt{n})^{(n+t\sqrt{n})}(1-t/\sqrt{n})^{(n-t\sqrt{n})}} \\ &= \frac{1}{(1+t/\sqrt{n})^n(1+t/\sqrt{n})^{t\sqrt{n}}(1-t/\sqrt{n})^n(1-t/\sqrt{n})^{-t\sqrt{n}}} \\ &= \frac{1}{(1-t^2/n)^n}\frac{(1-\frac{t^2}{t\sqrt{n}})^{t\sqrt{n}}}{(1+\frac{t^2}{t\sqrt{n}})^{t\sqrt{n}}} \to \frac{1}{e^{-t^2}}\frac{e^{-t^2}}{e^{t^2}} = e^{-t^2}. \end{aligned} \qquad A15.16$$

Putting this together, we see that

$$\frac{1}{C} \sum_{t \in T_{[a,b]}} \sqrt{\frac{2n}{n^2-t^2n}} \frac{1}{(1+t/\sqrt{n})^{(n+t\sqrt{n})}(1-t/\sqrt{n})^{(n-t\sqrt{n})}} \qquad A15.17$$

from equation A15.14 converges to

$$\frac{\sqrt{2}}{C}\frac{1}{\sqrt{n}}\sum_{t\in T_{[a,b]}} e^{-t^2}, \qquad A15.18$$

which is the desired Riemann sum. Thus as $n \to \infty$,

$$\frac{1}{2^{2n}}\sum_{k=a\sqrt{n}}^{b\sqrt{n}} \binom{2n}{n+k} \to \frac{\sqrt{2}}{C}\int_a^b e^{-t^2}\,dt. \qquad A15.19$$

We finally need to invoke a fact justified in section 4.11 (equation 4.11.73):

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}. \qquad A15.20$$

Since when $a = -\infty$ and $b = +\infty$ we must have

Exercise A15.1 gives another way to derive that $C = \sqrt{2\pi}$.

$$\frac{\sqrt{2}}{C}\sqrt{\pi} = \frac{\sqrt{2}}{C}\int_{-\infty}^{\infty} e^{-t^2}\,dt = 1, \qquad A15.21$$

we see that $C = \sqrt{2\pi}$, and finally

$$\underbrace{\frac{1}{2^{2n}}\sum_{k=a\sqrt{n}}^{b\sqrt{n}} \binom{2n}{n+k}}_{\substack{\text{prob.of having between}\\ n+a\sqrt{n}\text{ and }n+b\sqrt{n}\text{ heads}}} \quad \text{converges to} \quad \frac{1}{\sqrt{\pi}}\int_a^b e^{-t^2}\,dt. \quad \square \qquad A15.22$$

Exercises for section A.15

A15.1 This exercise sketches another way to find the constant in Stirling's formula. We will show that if there is a constant C such that

$$n! = C\sqrt{n}\left(\frac{n}{e}\right)^n (1 + o(1)),$$

as is proved in theorem A15.1, then $C = \sqrt{2\pi}$. The argument is fairly elementary, but not at all obvious. Let $c_n = \int_0^\pi \sin^n x\,dx$.

a. Show that $c_n < c_{n-1}$ for all $n = 1, 2, \dots$.

b. Show that for $n \geq 2$, we have $c_n = \frac{n-1}{n}c_{n-2}$. *Hint*: Write $\sin^n x = \sin x \sin^{n-1} x$ and integrate by parts.

c. Show that $c_0 = \pi$ and $c_1 = 2$, and use this and part b to show that

$$c_{2n} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{1}{2}\pi = \frac{(2n)!\pi}{2^{2n}(n!)^2}$$

$$c_{2n+1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{2}{3}2 = \frac{2^{2n}(n!)^2 2}{(2n+1)!}.$$

d. Use Stirling's formula with constant C to show that

$$c_{2n} = \frac{1}{C}\sqrt{\frac{2}{n}}\,\pi\Big(1 + o(1)\Big)$$

$$c_{2n+1} = \frac{C}{\sqrt{2n+1}}\Big(1 + o(1)\Big).$$