# Foreword

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I have long held a great admiration and appreciation for John Hamal Hubbard and his passionate engagement with mathematics. Hubbard has inspired me and many others. Passionate engagement is contagious. It shows through in his writing. This book develops a rich and interesting, interconnected body of mathematics that is also connected to many outside subjects. I commend it to you.

That's the short version. Here's a longer version:

Mathematics is a paradoxical, elusive subject, with the habit of appearing clear and straightforward, then zooming away and leaving us stranded in a blank haze.

Why?

It is easy to forget that mathematics is primarily a tool for human thought. Mathematical thought is far better defined and far more logical than everyday thought, and people can be fooled into thinking of mathematics as logical, formal, symbolic reasoning. But this is far from reality. Logic, formalization, and symbols can be very powerful tools for humans to use, but we are actually very poor at purely formal reasoning; computers are far better at formal computation and formal reasoning, but humans are far better mathematicians.

The most important thing about mathematics is how it resides in the human brain. Mathematics is not something we sense directly: it lives in our imagination and we sense it only indirectly. The choices of how it flows in our brains are not standard and automatic, and can be very sensitive to cues and context. Our minds depend on many interconnected special-purpose but powerful modules. We allocate everyday tasks to these various modules instinctively and subconsciously.

The term 'geometry', for instance, refers to a pattern of processing within our brains related to our spatial and visual senses, more than it refers to a separate content area of mathematics. One illustration of this is the concept of correlation between two measurements on a set, which is formally nearly identical with the concept of cosine of the angle between two vectors. The content is almost the same (for correlation, you first project to a hyperplane before measuring the cosine of the angle), but the human psychology is very different. Each mode of thinking has its own power, and ideally, people harness both modes of thought to work together. However, in formalized expositions, this psychological difference vanishes.

In the same way, any idea in mathematics can be thought about in many different ways, with competing advantages. When mathematics is explained, formalized and written down, there is a strong tendency to favor symbolic modes of thought at the expense of everything else, because symbols are easier to write and more standardized than other modes of reasoning. But when mathematics loses its connection to our minds, it dissolves into a haze.

I've loved to read all my life. I went to New College of Sarasota, Florida, a small college that was just starting up with a strong emphasis on independent study, so I ended up learning a good deal of mathematics by reading mathematics books. At that time, I prided myself in reading quickly. I was really amazed by my first encounters with serious mathematics textbooks. I was very interested and impressed by the quality of the reasoning, but it was quite hard to stay alert and focused. After a few experiences of reading a few pages only to discover that I really had no idea what I'd just read, I learned to drink lots of coffee, slow way down, and accept that I needed to read these books at 1/10th or 1/50th standard reading speed, pay attention to every single word and backtrack to look up all the obscure numbers of equations and theorems in order to follow the arguments. Even so, when something was "left to the reader", I generally left it as well. At the time, I could appreciate that the mathematics was an impressive intellectual edifice, and I could follow the steps of proofs. I assumed that such an elaborate buildup must be leading to a fantastic denouement, which I eagerly awaited – and waited, and waited.

It was only much later, after much of the mathematics I had studied had come alive for me that I came to appreciate how ineffective and denatured the standard $((\text{definition theorem proof})^n\ \text{remark})^m$ style is for communicating mathematics. When I reread some of these early texts, I was stunned by how well their formalism and indirection hid the motivation, the intuition and the multiple ways to think about their subjects: they were unwelcoming to the full human mind.

John Hubbard approaches mathematics with his whole mind.

If you page through the current book, you will see many intriguing figures. That is a first sign: figures are one of the most important ways to keep our thought processes going in our whole brains, rather than settling down into the linguistic, symbol-handling areas. Of course, the figures in your imagination are even more important. Geometric ideas can be conveyed with words and with symbols, sometimes more effectively than with pictures, but a lack of figures is a good indication of a lack of geometry.

Another important part of human thinking is the emotional aspect. In mathematics, what is intriguing, puzzling, interesting, surprising, boring, tedious, exciting is crucial; they are not incidental, they shape how we think.

Personally, my thinking was shaped by boredom: I develop intense urges to come up with 'easy' methods in order to avoid tedious computations that are opaque to me. Hubbard, a principal participant in the mathematics he is discussing, has done an excellent job in conveying the drama.

Teichmüller theory is an amazing subject, richly connected to geometry, topology, dynamics, analysis and algebra. I did not know this at the beginning of my career: as a topologist, I started out thinking of Teichmüller theory as an obscure branch of analysis irrelevant to my interests. My first encounter with Teichmüller theory was from the side. I was interested in some questions about isotopy classes of homeomorphisms of surfaces, and after struggling for quite a while, I finally proved a classification theorem for surface homeomorphisms, by first showing that set of all simple closed curves on a surface is parametrized as a subset of a Euclidean space. I was amazed to learn from Lipman Bers that this picture was implicit in the space of holomorphic quadratic differentials, by work of Hubbard and Masur. A few weeks after Bers invited me to give a some talks on surface homeomorphisms in his seminar at Columbia, I was even more amazed when Bers gave a new proof of my classification theorem by a method that was much simpler than my own, modulo principles of Teichmüller theory that had been developed decades earlier.

From this encounter I came to appreciate the beauty of Teichmüller theory, and of the close connections between 1-dimensional complex analysis and two and three-dimensional geometry and topology. A great deal of mathematics has been developed since that time and there are many active connections between geometry, topology, dynamics and Teichmüller theory, as indicated by the subtitle of this book.

Why is Teichmüller theory significant? All areas of mathematics tend to wax and wane, and Teichmüller theory in particular has gone through multiple cycles of popularity and unpopularity. There have been times when some (many?) mathematicians looked down on 1-dimensional complex analysis and on low-dimensional topology as special cases that are unrepresentative of general phenomena and unworthy of serious attention.

My view is that in mathematics, an internal test is the best gauge for the significance of a subject. If it is rich and interconnected and if it grabs your interest, then it is very likely to be become significant to you, even though in many cases you can't foresee how. Learning and absorbing mathematics is really a matter of adding software to your brain. We have strong and sophisticated mental filters designed to focus our attention away from what is unimportant and toward what is meaningful. If a mathematical topic seems rich, beautiful and interesting, that signals that it fills a significant mental role. If we allow ourselves to drink it in, it's highly likely to become useful, even if we don't have applications in mind.

Two-dimensional geometry is a special case, in many ways. As a start, there are infinitely many regular polygons. Regular polygons, unlike polyhedra in any higher dimension, are flexible. The group of isometries of the plane is solvable. The geometry of similarity in the plane is essentially the same as complex arithmetic. Topology in two dimensions is also a very special case. The topology of a closed oriented surface is measured by a simple invariant, the Euler characteristic. Every oriented surface is a complex 1-manifold, and in fact, any Riemannian metric on a surface determines a unique conformally equivalent complex structure. The list goes on and on: there are many phenomena that do not readily generalize to higher dimensions. This is a feature, not a bug: *because* two dimensions is a special case with many special features, two-dimensional topology, geometry and dynamics form an extraordinarily rich, beautiful and unique ecosystem that ends up being highly connected to a large array of other topics in mathematics and science.

I only wish that I had had access to a source of this caliber much earlier in my career.