7

The geometry of finite-dimensional Teichmüller spaces

Finite-dimensional Teichmüller spaces are the main focus of this book; they are the only Teichmüller spaces to appear in the later chapters. All that was said in Chapter 6 applies, but here these results have greater force.

7.1 Finite-dimensional Teichmüller spaces

When is a Teichmüller space finite dimensional?

Proposition 7.1.1 *The Teichmüller space $\mathcal{T}_X$ of a hyperbolic Riemann surface X is finite dimensional if and only if X is of finite type. If X is of genus g with n points removed, then $\mathcal{T}_X$ has complex dimension $3g-3+n$.*

Proof Recall (Definition 1.8.12) that a hyperbolic Riemann surface X is of finite type if it is isomorphic to a compact Riemann surface with a finite number of points removed. A hyperbolic Riemann surface X is of finite type if and only if it carries a finite geodesic multicurve such that all components of the complement are trousers (see Corollary 3.6.4).

Thus, if X is not of finite type, then either X contains an infinite multicurve or the ideal boundary $I(X)$ is nonempty, or both.

If there is an infinite multicurve Γ, then the lengths of any finite subset $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma$ can be varied arbitrarily, so that $\mathcal{T}_X$ is not finite dimensional.

If $I(X) \neq \emptyset$, you can choose an arbitrary quasisymmetric homeomorphism of $I(X)$ homotopic to the identity, and extend it to a quasiconformal homeomorphism of X. This provides an infinite-dimensional subset of $\mathcal{T}_S$.

If X is of finite type, of genus g with n punctures, then one approach to Proposition 7.1.1 is to claim that $Q^1(X)$ (the space of integrable quadratic differentials on X) is finite dimensional, in fact has finite dimension $3g-3+n$ by the Riemann-Roch theorem (Theorem A10.0.1).

Let us try to find this number with a more topological approach; we will expand on this in Section 7.6. First, a straightforward calculation using the Euler characteristic shows that a maximal multicurve Γ has $3g-3+n$ components. The resulting trousers are then determined by the lengths of the curves, i.e., $3g-3+n$ positive real numbers. To determine a Riemann surface, we have to specify how to glue the trousers together. Clearly we can glue the trousers together if the boundary components corresponding to the same element of Γ have the same lengths. In that case, we can rotate one side of each γ_i with respect to the other, giving $3g-3+n$ more real parameters; this is illustrated by Figure 7.1.1. □

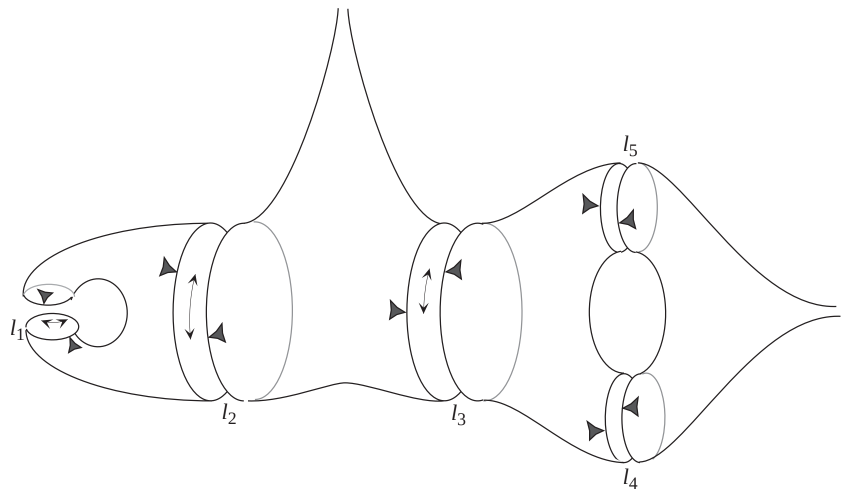

FIGURE 7.1.1 A surface of genus $g = 2$ with $n = 2$ punctures is decomposed into four trousers. The trousers are completely specified by the lengths $l_1, \dots, l_5$ of their boundary components, but to assemble them, we have to know at what angle the boundary components should be sewn together. This provides five more parameters, giving 10 in all. Indeed, $6g-6+2n = 12-6+4 = 10$.

When X is not of finite type, the Teichmüller space $\mathcal{T}_X$ depends in a rather delicate way on the complex structure of X.

Example 7.1.2 (Homeomorphic Riemann surfaces in different Teichmüller spaces) Let $X := \mathbb{C} - \big(\{\, 2^n \mid n \in \mathbb{Z}\,\} \cup \{0\}\big)$, and let γ_0 be the geodesic in the homotopy class of the circle of radius 3/2 centered at the origin. Let $\gamma_n := 2^n \gamma_0,\ n \in \mathbb{Z}$; since multiplication by 2 is an automorphism of X, these curves are all geodesics; together they give a trouser decomposition of X. It should be clear that we can put a Beltrami form on each trouser so as to make the lengths of these geodesics any sequence $(l_n)_{n\in\mathbb{Z}}$ of positive numbers we like. But if the sequence $(\ln l_n)_{n\in\mathbb{Z}}$ is not bounded, the corresponding Riemann surface will not belong to the same Teichmüller space as X. △

7.2 Teichmüller's theorem

In Corollary 6.7.2 we saw that Teichmüller space $\mathcal{T}_S$ is contractible. Now we will construct an explicit homeomorphism of Teichmüller space with a ball, when the Teichmüller space is finite dimensional. This result gives much more than contractibility: it describes the geodesic discs in Teichmüller space for the Teichmüller metric.

Let X be a Riemann surface of finite type, and let $q \in Q^1(X)$ be a holomorphic quadratic differential on X that does not vanish identically. Then $\overline{q}/|q|$ is an infinitesimal Beltrami form with L^∞-norm 1. In particular, if $B(X) \subset Q^1(X)$ is the open unit ball and $q \in B(X)$, then the Beltrami form

$$\|q\|_1 \frac{\overline{q}}{|q|} \tag{7.2.1}$$

defines a new complex structure on X; recall that $|q|$ denotes the element of area; see equation 5.3.3. We will denote by X_q the Riemann surface with underlying quasiconformal surface $qc(X)$; with this complex structure, analytic functions on X_q are solutions of the equation

$$\bar{\partial}\zeta = \|q\|_1 \frac{\overline{q}}{|q|}\, \partial\zeta. \tag{7.2.2}$$

The Riemann surface X_q marked by the identity $X \to X_q$ is an element of the Teichmüller space $\mathcal{T}_X$; we will denote this element by $F(q)$.

Theorem 7.2.1 (Teichmüller's theorem on contractibility) *Let X be a Riemann surface of finite type. Then the map $F\colon B(X) \to \mathcal{T}_X$ is a homeomorphism.*

Proof All the difficult work was done in Theorem 5.3.8. We will show that F is injective, continuous, and proper. Continuity follows immediately from the continuity of solutions of the Beltrami equation: if $\mathcal{M}(X)$ is given the L^1 topology (but not the L^∞ topology), then clearly the map

$$q \mapsto \|q\|_1 \frac{\overline{q}}{|q|} \tag{7.2.3}$$

is continuous from $B(X)$ to $\mathcal{M}(X)$.

To see that F is injective and proper, observe that X_q naturally carries the holomorphic quadratic differential q', which can be written as follows: if $z = x + iy$ is a natural coordinate for q in some subset $U \subset X$, then $q' = (dx + i\,dy/K)^2$ in U.

Exercise 7.2.2 Show that q' is indeed holomorphic on X_q. $\diamondsuit$

The identity map $X \to X_q$ is a Teichmüller mapping from (X, q) to (X_q, q'); see Definition 5.3.6. As such, it is the unique quasiconformal map $f\colon X \to X_q$ in the homotopy class of the identity that minimizes the deformation of the complex structure. Thus if $F(q_1) = F(q_2)$, then there exists an analytic map $\alpha\colon X_{q_1} \to X_{q_2}$ homotopic to the identity (see equation 6.4.1):

$$\begin{array}{ccc} & & X_{q_1} \\ & \overset{\text{id}}{\nearrow} & \\ X & & \downarrow \alpha \\ & \underset{\text{id}}{\searrow} & \\ & & X_{q_2} \end{array} \qquad 7.2.4$$

Both $\text{id}\colon X \to X_{q_2}$ and $\alpha \circ \text{id}\colon X \to X_{q_2}$ deform the complex structure the same amount, so $\alpha = id$ and $q_1 = q_2$.

Thus F is an injective continuous map between manifolds of the same dimension, so it is a homeomorphism to its image. This is where we are using the fact that our Teichmüller spaces are finite dimensional.

We still need to see that F is proper. But if a sequence (q_n) converges in the unit ball $B(X)$, then the $F(q_n)$ remain a bounded distance from $F(0)$, hence they remain in a compact subset of $\mathcal{T}_X$. So F is proper. □

One consequence of Theorem 7.2.1 is that Teichmüller maps (see Definition 5.3.6) minimize the distortion of the complex structure in their homotopy classes, and hence realize the infimum in the definition of the Teichmüller distance in equation 6.4.2:

Corollary 7.2.3 *Let X be a Riemann surface of finite type and let $f\colon X \to X$ be a homeomorphism. Denote by $\sigma_f(X) \in \mathcal{T}_X$ the pair $(X\,,\, f\colon X \to X)$. Then there exists a unique Teichmüller mapping $g\colon X \to X$ homotopic to f, and*

$$d\big(X, \sigma_f(X)\big) = \ln K(g). \qquad 7.2.5$$

7.3 The Mumford compactness theorem

We saw in Section 3.8 that short geodesics are surrounded by long collars, with a geometry that is completely understood. The object of this section is to show that the remainder of the Riemann surface has *bounded geometry.*

Recall Definitions 6.4.13 and 6.4.14 of the Teichmüller modular group $\text{MCG}(S)$ and moduli space $\text{Moduli}(S)$. Let $\text{Moduli}_c(S) \subset \text{Moduli}(S)$ consist of Riemann surfaces whose simple closed geodesics all have length at least c.

It should be clear that Moduli(S) depends on S only through its homeomorphism type, i.e., its genus. Another way of saying this is that an element of Moduli(S) is a Riemann surface homeomorphic to S, but without any distinguished homeomorphism.

Theorem 7.3.1 (Mumford compactness theorem) *Let S be a compact Riemann surface of genus $g \geq 2$. For every $c > 0$, the space* $\text{Moduli}_c(S)$ *is compact.*

PROOF Let S have genus g, and hence hyperbolic area $4\pi(g-1)$ by the Gauss-Bonnet theorem. For every Riemann surface $X \in \text{Moduli}_c(S)$ and for every $x \in X$, the closed disc of radius $c/2$ centered at x is embedded. Consider the space $R_c(S)$ consisting of pairs (X, Γ), where $X \in \text{Moduli}_c(S)$ is a Riemann surface, and $\Gamma := \{\gamma_i, i = 1, \dots k\}$ is a maximal collection of isometric embeddings $\gamma_i : D_{c/4} \to X$ of open hyperbolic discs of radius $c/4$ with disjoint images (a collection of disjoint discs in X is called a *disc packing*). See Figure 7.3.1.

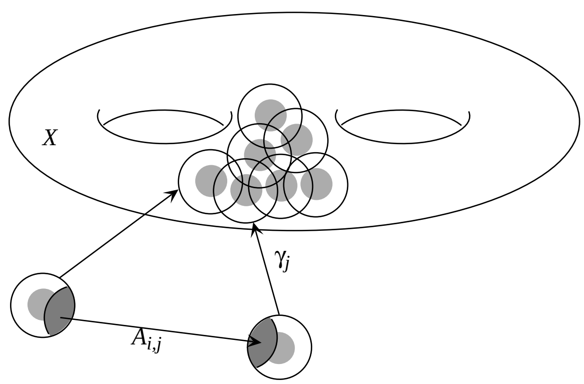

FIGURE 7.3.1 A Riemann surface is represented, together with a family of shaded discs of radius $c/4$, all disjoint. If this family is maximal, then the family of concentric discs of radius $c/2$ covers X. The discs are images of the standard disc centered at the origin (in the disc model of $\mathbb{H}^2$) by isometries γ_i. If the images of two of these discs intersect, then there is a Möbius transformation $A_{i,j}$ such that $\gamma_i = \gamma_j \circ A_{i,j}$.

When a packing of discs of radius $c/4$ is maximal, then the concentric open discs of radius $c/2$ cover X. Indeed, if a point is not in their union, then the disc of radius $c/4$ centered at that point is disjoint from all the other discs of radius $c/4$, so the original collection of disjoint discs was not maximal.