5

Preliminaries to Teichmüller theory

In this chapter we present a number of results that play an essential role in our theory of Teichmüller spaces:

1. The Douady-Earle extension theorem and the associated reflection theorem, Section 5.1.
2. Slodkowski's theorem on extensions of holomorphic motions, Section 5.2.
3. Teichmüller's theorem on extremal mappings between Riemann surfaces, Section 5.3.
4. Several results concerning spaces of quadratic differentials in Section 5.4, more particularly, the duality theorem, Theorem 5.4.12.

Each is of great interest in its own right.

5.1 The Douady-Earle extension

Theorem 4.9.5 says that every quasisymmetric homeomorphism $f:\mathbb{R}\to\mathbb{R}$ extends to a quasiconformal homeomorphism $\mathbf{H}\to\mathbf{H}$. In this section we describe an especially nice such extension, due to Douady and Earle [30], which has a crucial naturality property. At the end of this section we will deduce from it a reflection theorem (Theorem 5.1.13) due to Earle and Nag [46].

It is more convenient to deal with quasisymmetric maps $f:S^1\to S^1$. We already know (Definition 4.5.13) that a map $f:X\to Y$ for any metric spaces X and Y is L-quasisymmetric if there exists a homeomorphism $\eta:[0,\infty)\to[0,\infty)$ such that for any distinct points $a,b,c\in X$ we have

$$\left|\frac{f(a)-f(b)}{f(a)-f(c)}\right|\le\eta\left(\left|\frac{a-b}{a-c}\right|\right). \tag{5.1.1}$$

We also have Definition 4.9.3 of $\mathbb{R}$-quasisymmetric maps. The following exercise asks you to show that they are equivalent. It is not very different from the equivalence of parts 2 and 3 in Theorem 4.9.19.

Exercise 5.1.1 Let $f:S^1\to S^1$ be a homeomorphism. Show that the following two conditions are equivalent:

1. f is L-quasisymmetric with modulus η.
2. There exists a constant M such that for any $a\in S^1$, if the analytic isomorphism $\gamma_1:\mathbf{D}\to\mathbf{H}$ maps ∞ to a and the analytic isomorphism

$\gamma_2 : \mathbf{D} \to \mathbf{H}$ maps $f(a)$ to ∞, then the function $g : \mathbb{R} \to \mathbb{R}$ given by $g := \gamma_2 \circ f \circ \gamma_1$ satisfies

$$\frac{1}{M} \le \frac{g(x+t) - g(x)}{g(x) - g(x-t)} \le M. \qquad \diamondsuit \qquad 5.1.2$$

By extension, we will say that a homeomorphism $f : S^1 \to S^1$ is $\mathbb{R}$-*quasisymmetric* with modulus M if it satisfies the second condition of Exercise 5.1.1 with constant M. Let $\mathbf{QS}_M(S^1)$ denote the space of homeomorphisms $f : S^1 \to S^1$ that are $\mathbb{R}$-quasisymmetric with modulus M; recall from Definition 4.7.3 that $\mathbf{QC}_K(\mathbf{D})$ denotes the space of K-quasiconformal homeomorphisms $\mathbf{D} \to \mathbf{D}$.

Theorem 5.1.2 (The Douady-Earle extension theorem) *For any $M \ge 1$, there exist $K \ge 1$ and a map $\Phi : \mathbf{QS}_M(S^1) \to \mathbf{QC}_K(\mathbf{D})$ such that the K-quasiconformal map $\Phi(f)$ extends f and for every $\gamma_1, \gamma_2 \in \operatorname{Aut}\mathbf{D}$,*

$$\Phi(\gamma_1 \circ f \circ \gamma_2) = \gamma_1 \circ \Phi(f) \circ \gamma_2. \qquad 5.1.3$$

The proof will be completed by the end of the section. To lighten notation, we will denote $\Phi(f)$ by $\hat{f}$.

The conformal barycenter

In this section we will use measures (denoted μ) throughout. Beltrami forms will appear only in the last part, and then not explicitly. Recall that an atom of a measure μ is a point with mass: a point p such that $\mu(\{p\}) \ge 0$.

Below, γ_* denotes "push forward" by γ, in whatever setting is appropriate: $\gamma_*\mu$ is the push forward of the measure μ (direct images of measures are always well defined), and $\gamma_*\vec{\xi}$ is the push forward of the vector field $\vec{\xi}$, only well defined because γ is an isomorphism.

Proposition and Definition 5.1.3 (Conformal barycenter)

1. *There exists a unique mapping $\mu \mapsto \vec{\xi}_\mu$ from the space of probability measures on S^1 to the space of C^∞ vector fields on $\mathbf{D}$, having the following two properties:*
   a. $\vec{\xi}_\mu(0) = \int_{S^1} \zeta \; \mu(d\zeta)$.
   b. *For every $\gamma \in \operatorname{Aut}\mathbf{D}$, we have $\vec{\xi}_{\gamma_*\mu} = \gamma_*\vec{\xi}_\mu$.*
2. *If μ has no atoms, then $\vec{\xi}$ has a unique 0 in the interior of $\mathbf{D}$, called the conformal barycenter of μ and denoted $B(\mu)$.*

PROOF 1. Observe that if γ is a rotation, then our formula for $\vec{\xi}_\mu(0)$ guarantees that

$$\gamma_* \vec{\xi}_\mu(0) = \vec{\xi}_{\gamma_*\mu}(0). \qquad 5.1.4$$

Part 1 follows: for any point $a \in \mathbf{D}$, move a to 0 by an element $\gamma \in \operatorname{Aut}\mathbf{D}$ and define

$$\vec{\xi}_\mu(a) := [D\gamma(a)]^{-1} \vec{\xi}_{\gamma_*\mu}(0). \qquad 5.1.5$$

Any $\gamma_1 \in \operatorname{Aut}\mathbf{D}$ with $\gamma_1(a) = 0$ can be written $\gamma_1 = \delta \circ \gamma$, where δ is a rotation, so

$$\begin{aligned} [D\gamma_1(a)]^{-1}(\vec{\xi}_{(\gamma_1)_*\mu}(0)) &= \Big([D\delta(\gamma(a))][D\gamma(a)]\Big)^{-1}\big(\delta_*\vec{\xi}_{\gamma_*\mu}(0)\big) \\ &= [D\gamma(a)]^{-1}\big(\delta_*^{-1}\delta_*\vec{\xi}_{\gamma_*\mu}(0)\big). \end{aligned} \qquad 5.1.6$$

Thus the vector field $\vec{\xi}_\mu$ is well defined. It is then straightforward to give an explicit formula for $\vec{\xi}_\mu$:

$$\vec{\xi}_\mu(z) = \big(1 - |z|^2\big) \int_{S^1} \frac{\zeta - z}{1 - \overline{z}\zeta}\, \mu(d\zeta). \qquad 5.1.7$$

(When we integrate with respect to a measure μ and need to specify the variable of integration x, we write $\mu(dx)$ – "μ measures little pieces of x" – rather than the more standard $d\mu(x)$.)

2. We can use equation 5.1.7 to compute the Jacobian of $\vec{\xi}_\mu$ at 0. First, compute

$$\begin{aligned} \vec{\xi}_\mu(z) &= \big(1 - |z|^2\big) \int_{S^1} (\zeta - z)(1 + \overline{z}\zeta + (\overline{z}\zeta)^2 + \cdots)\mu(d\zeta) \\ &= \vec{\xi}_\mu(0) - z + \overline{z}\int_{S^1} \zeta^2 \mu(d\zeta) + o(|z|). \end{aligned} \qquad 5.1.8$$

This gives the partial derivatives

$$\frac{\partial \vec{\xi}_\mu}{\partial z}(0) = -1, \quad \frac{\partial \vec{\xi}_\mu}{\partial \overline{z}}(0) = \int_{S^1} \zeta^2 \mu(d\zeta). \qquad 5.1.9$$

Finally (using Definition 4.1.5), the Jacobian is

$$\begin{aligned} \left|\frac{\partial \vec{\xi}_\mu}{\partial z}(0)\right|^2 - \left|\frac{\partial \vec{\xi}_\mu}{\partial \overline{z}}(0)\right|^2 &= 1 - \int_{S^1 \times S^1} \zeta_1^2\, \overline{\zeta_2}^{\,2}\, \mu(d\zeta_1)\, \mu(d\zeta_2) \\ &= \frac{1}{2}\int_{S^1 \times S^1} \left|\zeta_1^2 - \zeta_2^2\right|^2 \mu(d\zeta_1)\mu(d\zeta_2), \end{aligned} \qquad 5.1.10$$

which is strictly positive, since

$$\int_{S^1} |\zeta|^4 \mu(d\zeta) = \int_{S^1} \mu(d\zeta) = 1. \qquad 5.1.11$$

In particular, all the zeros of $\vec{\xi}_\mu$ have index 1.

Moreover, we can easily see that $\vec{\xi}_\mu$ points inward near the boundary $\partial\mathbf{D}$. Indeed, if $|z|$ is close to 1 and $\gamma(\zeta) = (\zeta - z)/(1 - \overline{z}\zeta)$, then $\gamma_*\mu$ is approximately a unit mass at $-z/|z|$. Thus $\vec{\xi}_{\gamma_*\mu}(0)$ is approximately $-z\partial/\partial z$, and this vector points inwards when moved back to z. The sum of the indices of the zeros of $\vec{\xi}_\mu$ is 1 by the Poincaré-Hopf index theorem, so $\vec{\xi}_\mu$ has a unique zero. □

Remark 5.1.4 The proof usually works even if μ has atoms; it fails only when μ has an atom with weight $\geq 1/2$. In that case, the proof fails at two places: first, the integral, which should be positive to see that the index is 1, can vanish; second, the vector field does not point inwards near the boundary. But the conformal barycenter exists anyway in $\overline{\mathbf{D}}$, and is the atom of weight $\geq 1/2$, except in the case where there are two atoms at distinct points of weight 1/2; in that case the vector field $\vec{\xi}_\mu$ vanishes on the geodesic joining the points, and there is no conformal barycenter. △

Proof of the Douady-Earle extension

To every point $z \in \mathbf{D}$ we can associate the *harmonic measure* η_z of z on S^1:

$$\eta_z := \frac{1}{2\pi} \frac{1 - |z|^2}{|\overline{z}\zeta + 1|^2} |d\zeta|. \qquad 5.1.12$$

As illustrated by Figure 5.1.1, this harmonic measure associates to every arc the normalized angle under which it is seen from z using the hyperbolic metric.

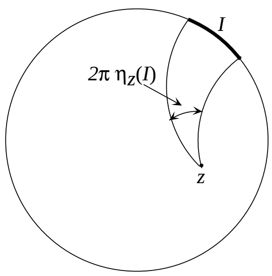

FIGURE 5.1.1 To find the harmonic measure $\eta_z(I)$, draw the hyperbolic geodesics from a point $z \in \mathbf{D}$ to the endpoints of an interval I.

Definition 5.1.5 (Douady-Earle extension) Let B denote the conformal barycenter defined in Proposition and Definition 5.1.3. The *Douady-Earle extension* of a continuous mapping $f : S^1 \to S^1$ is the map $\Phi(f) : \mathbf{D} \to \mathbf{D}$ given by $\Phi(f)(z) := B(f_*\eta_z)$.

To lighten notation, we will usually denote $\Phi(f)$ by $\hat{f}$. The map Φ is defined for all continuous maps $f : S^1 \to S^1$ if $f_*\eta_z$ has no atoms. But if f

collapses some interval I to ζ, then $f_*\eta_z$ will have an atom of weight $\geq 1/2$ at ζ for z in the convex hull $\hat{I}$ of I, so that $\hat{f}$ is still defined but maps $\hat{I}$ to ζ. The bad case, where $f_*\eta_z$ is the sum of two atoms of weight $1/2$, does not occur: a continuous map cannot collapse the circle to two distinct points. So Φ is defined on the space $\mathcal{C}(S^1, S^1)$ of continuous maps from the circle to the circle, but we must consider $\hat{f}$ as a map $\mathbf{D} \to \overline{\mathbf{D}}$.

Proposition 5.1.6 (Properties of the Douady-Earle extension)

1. *The Douady-Earle extension $\hat{f}$ extends continuously to $\overline{\mathbf{D}}$, agrees with f on the boundary, and is real analytic on $\hat{f}^{-1}(\mathbf{D})$.*
2. *The map $\Phi : \mathcal{C}(S^1, S^1) \to \mathcal{C}(\overline{\mathbf{D}}, \overline{\mathbf{D}})$ is continuous, using the topology of uniform convergence on the circle in the domain and the topology of uniform convergence on $\overline{\mathbf{D}}$ in the codomain.*
3. *The map $\hat{f}$ has the desired naturality: if $\gamma_1,\ \gamma_2 \in \operatorname{Aut}\mathbf{D}$, then*

$$\gamma_1 \circ \hat{f} \circ \gamma_2 = \widehat{\gamma_1 \circ f \circ \gamma_2}. \tag{5.1.13}$$

The restriction "on $\hat{f}^{-1}(\mathbf{D})$" in part 1 avoids the boundaries $\partial\hat{I}$ of intervals I collapsed to points.

Proof Parts 2 and 3 are obvious. The first statements of part 1, that $\hat{f}$ extends continuously to $\overline{\mathbf{D}}$ and that it agrees with f on the boundary, follow from the fact that η_z tends to the δ-mass at x when z tends to a point x in the boundary of the disc. Now we will see that $\hat{f}$ is real analytic on $\hat{f}^{-1}(\mathbf{D})$, i.e., at points where $\hat{f}(z)$ is the zero of the vector field $\xi_{f_*\eta_z}$. The graph of $\hat{f}$, i.e., the locus of equation $\hat{f}(z) = w$, is (by equations 5.1.7 and 5.1.12) defined implicitly by the equation $F(z, w) = 0$, where

$$F(z,w) = \frac{1}{2\pi}\int_{S^1} \frac{f(\zeta) - w}{1 - \overline{w} f(\zeta)}\, \frac{1 - |\zeta|^2}{|z - \zeta|^2}\, |d\zeta|. \tag{5.1.14}$$

Since F is real analytic, so is $\hat{f}$. Note that we already know, from the uniqueness of the conformal barycenter, that the derivative of F with respect to w is a non-singular 2×2 matrix; but in any case we will later need to compute the derivative of $\hat{f}$, which requires the inverse of this matrix; this derivative is computed in equation 5.1.17. □ Proposition 5.1.6

So far we have proved all of the Douady-Earle extension theorem except for the statement that the extended map $\hat{f}$ is K-quasiconformal, where K depends only on the quasisymmetric modulus M of f.

Proposition 5.1.7 *If $f : S^1 \to S^1$ is a homeomorphism, then $\hat{f}$ is a diffeomorphism.*