4

Quasiconformal maps and the mapping theorem

Quasiconformal maps form a branch of complex analysis. I found the subject difficult to learn, mainly because I had a hard time appreciating how smooth the maps are. They are somehow rather magical, with properties that seem contradictory. They are smooth enough that much of calculus holds: the chain rule and the integral formulas for lengths and areas. They are rough enough that conjugating by them can change derivatives at fixed points. They are homeomorphisms, but have an affine structure: there are barycenters of quasiconformal mappings, and canonical "straight lines" joining pairs.

Our treatment is somewhat different from the standard one: it is strongly colored by a prejudice in favor of *soft analysis wherever possible*. Thus we avoid the words *almost everywhere* when we can, and more generally we avoid evaluating functions unless they are continuous: measurable functions should appear only under integral signs. Distributions are in, differentiability a.e. is out. Differential forms are in, densities are out. Approximations by C^1 functions are in, absolute continuity on lines is out.

4.1 Two analytic definitions

There are several possible definitions of quasiconformal mappings, and it is not so easy to see that they are equivalent. In this section we will give the best definition for our present purposes; in Section 4.5 we will give another and will propose three more in exercises.

The great virtue of Definition 4.1.1 below is that it is well adapted to the proof of the mapping theorem, Theorem 4.6.1. However, it has drawbacks: although inverses and compositions of quasiconformal maps are quasiconformal, this does not follow easily from this analytic definition. Nor does this definition make it easy to check whether various explicit mappings are quasiconformal.

Remark Definition 4.1.1 involves distributional partial derivatives, often called *weak derivatives*. I dislike this misleading name, which suggests that a weak derivative carries inadequate information. Exactly the opposite is true: distributional derivatives carry *all* the information that a derivative should carry, unlike derivatives almost everywhere, which often overlook essential features. See Example 4.1.8 for a striking illustration. △

Definition 4.1.1 (Quasiconformal map: first analytic definition) Let U, V be open subsets of $\mathbb{C}$, take $K \geq 1$, and set $k := (K-1)/(K+1)$, so that $0 \leq k < 1$. A mapping $f : U \to V$ is K-*quasiconformal* if it is a homeomorphism whose distributional partial derivatives are in L^2_{loc} (locally in L^2) and satisfy

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right| \qquad 4.1.1$$

in L^2_{loc}, i.e., almost everywhere.

A map is *quasiconformal* if it is K-quasiconformal for some K.

Definition 4.1.2 (Quasiconformal constant) The smallest K such that f is K-quasiconformal is called the *quasiconformal constant* of f, denoted $K(f)$.

The quasiconformal constant is sometimes called the *quasiconformal norm* and sometimes the *quasiconformal dilatation*.

The constant K measures how near a mapping is to being conformal, i.e., analytic; the closer K is to 1, the more nearly conformal a K-quasiconformal map is. This is not the only possible definition of what is means to be "nearly conformal", but it is the most useful one, because good theorems are available for it.

The meaning of inequality 4.1.1 is best understood if $f \in C^1(U)$. Then the derivative $[Df(z_0)]$ is an $\mathbb{R}$-linear map, given by the Jacobian matrix, but it is easier to use complex notation:

$$[Df(z_0)](u) = \frac{\partial f}{\partial z}(z_0)u + \frac{\partial f}{\partial \bar{z}}(z_0)\bar{u}. \qquad 4.1.2$$

If we write a real linear transformation $T : \mathbb{C} \to \mathbb{C}$ as $T(u) = au + b\bar{u}$, so that $a = \frac{\partial T}{\partial u}$ and $b = \frac{\partial T}{\partial \bar{u}}$, then we will see below that the determinant and norm of T are given by the important formulas

$$\det T = |a|^2 - |b|^2 \quad , \quad \|T\| = |a| + |b|. \qquad 4.1.3$$

Remark 4.1.3 It follows from equation 4.1.3 that if T preserves orientation, then $|\frac{\partial T}{\partial \bar{u}}| < |\frac{\partial T}{\partial u}|$, and if T reverses orientation, then $|\frac{\partial T}{\partial \bar{u}}| > |\frac{\partial T}{\partial u}|$. (If the two sides are equal, then T is not an isomorphism, since it is neither orientation preserving nor orientation reversing.) △

Both formulas follow from computing the inverse image of the unit circle, i.e., from computing the real curve of equation $|T(u)| = 1$. Write

$$u := re^{i\theta}, \quad a := |a|e^{i\alpha}, \quad \text{and} \quad b := |b|e^{i\beta}. \qquad 4.1.4$$

The equation $|T(u)| = 1$ becomes in polar coordinates

$$\left|(|a| + |b|)\cos\left(\theta + \frac{\alpha - \beta}{2}\right) + i(|a| - |b|)\sin\left(\theta + \frac{\alpha - \beta}{2}\right)\right| = \frac{1}{r}. \qquad 4.1.5$$

This is the equation of an ellipse, with

– minor axis at polar angle $\dfrac{\beta - \alpha}{2}$ of semi-length $\dfrac{1}{|a| + |b|}$, and

– major axis at polar angle $\dfrac{\beta - \alpha + \pi}{2}$ of semi-length $\dfrac{1}{||a| - |b||}$.

This is illustrated in Figure 4.1.1. In particular, $\|T\| = |a| + |b|$ (the inverse of the semi-length of the minor axis), and $\det T = |a|^2 - |b|^2$ (up to sign, the ratio of the area of the unit circle to the area of its preimage).

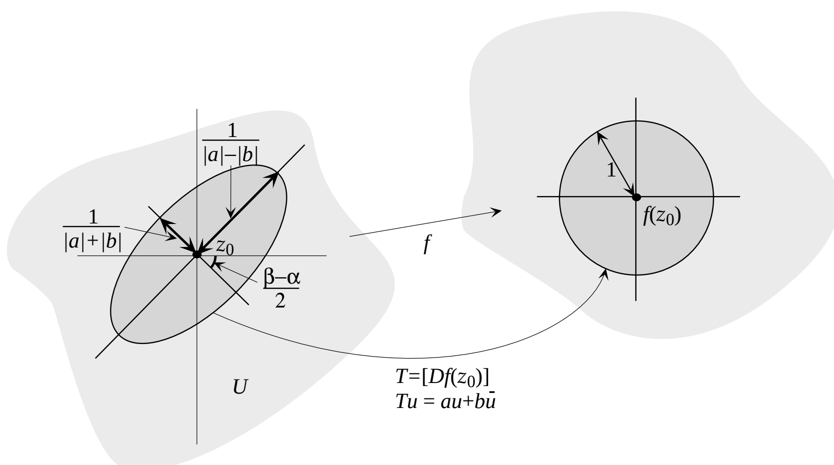

FIGURE 4.1.1 If f is K-quasiconformal and of class C^1 (so that its derivative exists), then its derivative at z_0 takes the the ellipse on the left to the unit circle on the right.

Finally, the ratio of the axes of the ellipse is

$$\frac{|a| + |b|}{|a| - |b|} \leq \frac{1 + k}{1 - k} = K. \qquad 4.1.6$$

Now set $a := \frac{\partial f}{\partial z}(z_0)$, $b := \frac{\partial f}{\partial \bar{z}}(z_0)$ and write equation 4.1.2 in the form $[Df(z_0)]u = au + b\bar{u}$. Then if $f \in C^1(U)$ is K-quasiconformal, the condition $0 \leq k < 1$ in Definition 4.1.1 implies that $\det[Df(z_0)]$ is everywhere positive, so that f preserves orientation; see Remark 4.1.3.

This gives the explanation we were after: a K-quasiconformal mapping of class C^1 is an orientation-preserving diffeomorphism whose derivative maps infinitesimal circles to infinitesimal ellipses with eccentricity at most K (i.e., the ratio of the lengths of the axes of the ellipses is bounded by K).

Sometimes we know $f : U \to V$ in real terms:

$$f(x + iy) = u(x, y) + iv(x, y). \qquad 4.1.7$$

Computing the operator norm $\|[Df]\|$ is then a bit unpleasant; the easy thing to compute is

$$|[Df]|^2 := \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2. \qquad 4.1.8$$

Exercise 4.1.4 Show that

$$K(x + iy) + \frac{1}{K(x + iy)} = \frac{|[Df(x, y)]|^2}{\operatorname{Jac} f(x, y)}. \qquad \diamondsuit \qquad 4.1.9$$

Note that if $U, V \subset \mathbb{C}$ are open and $f : U \to V$ is a continuous map whose distributional derivatives are locally in L^2, then

$$\operatorname{Jac} f = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar z}\right|^2 \quad \text{and} \quad \|[Df]\|^2 = \left(\left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \bar z}\right|\right)^2 \qquad 4.1.10$$

are locally in L^1.

Thus Definition 4.1.1 can be restated as follows:

Definition 4.1.5 (Quasiconformal map: 2nd analytic definition) Let U, V be open subsets of $\mathbb{C}$ and take $K \geq 1$. A map $f : U \to V$ is *K-quasiconformal* if

1. it is a homeomorphism,
2. its distributional partial derivatives are locally in L^2, and
3. its distributional partial derivatives satisfy

$$\operatorname{Jac} f \geq \frac{1}{K}\|[Df]\|^2 \quad \text{locally in } L^1. \qquad 4.1.11$$

Note that f is necessarily orientation preserving, since the Jacobian is positive by part 3.

Inequalities 4.1.1 and 4.1.11 would not make sense if distributional partial derivatives were simply distributions. They would make sense if the derivatives were only required to exist a.e. and to be in L^2. Some authors mistakenly use this definition of quasiconformal mapping. It is not a useful definition, because the resulting maps do not have the desired properties. In particular, Weyl's lemma would be false, as shown in Example 4.1.8.

Theorem 4.1.6 (Weyl's lemma) *If $U \subset \mathbb{C}$ is open, and $f : U \to \mathbb{C}$ is a distribution in U satisfying $\partial f / \partial \bar{z} = 0$, then f is an analytic function on U.*

Proof Choose $r > 0$, let $D_r(z)$ be the disc of radius r centered at z, and let φ_ϵ be a family of test functions with support in $D_r(0)$ and tending to the delta function as $\epsilon \to 0$. Then the convolutions $f_\epsilon = f * \varphi_\epsilon$ are C^∞ functions on $U_r := \{ z \in U \mid D_r(z) \subset U \}$, and the f_ϵ satisfy $\partial f_\epsilon / \partial \bar{z} = 0$. (This is not true for the function of Example 4.1.8. It is essential that it is the distributional derivative that vanishes.) Therefore each f_ϵ is an analytic function on U_r.

We want to show that the f_ϵ converge uniformly on compact subsets as $\epsilon \to 0$; for this we need a slight variation on the Cauchy integral formula. Choose $r_1 < r_2$ and a C^∞ function η with support in (r_1, r_2) with $\int_{r_1}^{r_2} \eta(r)\, dr = 1$. Then the equation

$$f_\epsilon(z) = \frac{1}{2\pi i} \int_{r_1}^{r_2} \int_0^{2\pi} \frac{f_\epsilon(z_0 + re^{i\theta})}{z - (z_0 + re^{i\theta})} \eta(r)\, d\theta\, dr \qquad 4.1.12$$

is true in the disc of radius r_1 around any point $z_0 \in U_{r+r_2}$. In equation 4.1.12, for each fixed z, the distribution f_ϵ is evaluated on the fixed test function $\dfrac{\eta(r)}{z - (z_0 + re^{i\theta})}$, so it converges as $\epsilon \to 0$, giving f a value at every point. Since the test functions vary continuously as functions of z, the function f is continuous. Using an appropriate variant of the Cauchy integral formula, it is not much harder to show that the derivative exists and is continuous. □

Corollary 4.1.7 *A 1-quasiconformal mapping is analytic, in fact, it is a conformal mapping, since it is a homeomorphism.*

Proof A 1-quasiconformal mapping satisfies equation 4.1.1 with $k = 0$, i.e., it satisfies the hypothesis of Weyl's lemma. □

The following example shows how badly behaved a homeomorphism can be when it is only differentiable almost everywhere, with the derivatives satisfying inequalities 4.1.1 and 4.1.11. *This example should be kept in mind throughout this chapter.* In some sense the whole chapter is a fight against it: we are constantly worried that some part of the distributional derivative is hiding in a set of measure 0.

Example 4.1.8 (A homeomorphism of $\mathbb{R}^2$ that is not quasiconformal) Let the function $\eta : \mathbb{R} \to \mathbb{R}$ be the standard "devil's staircase": the unique nondecreasing function such that