3

Hyperbolic geometry of Riemann surfaces

By Theorem 1.8.8, all hyperbolic Riemann surfaces inherit the geometry of the hyperbolic plane. How this geometry interacts with the topology of a Riemann surface is a complicated business, and beginning with Section 3.2, the material will become more demanding. Since this book is largely devoted to the study of Riemann surfaces, a careful study of this interaction is of central interest and underlies most of the remainder of the book.

3.1 Fuchsian groups

We saw in Proposition 1.8.14 that torsion-free Fuchsian groups and hyperbolic Riemann surfaces are essentially the same subject. Most such groups and most such surfaces are complicated objects: usually, a Fuchsian group is at least as complicated as a free group on two generators.

However, in a few exceptional cases Fuchsian groups are not complicated, whether they have torsion or not. Such groups are called *elementary*; we classify them in parts 1–3 of Proposition 3.1.2. Part 4 concerns the complicated case – the one that really interests us.

Notation 3.1.1 If A is a subset of a group G, we denote by $\langle A \rangle$ the subgroup generated by A.

Proposition 3.1.2 (Fuchsian groups) *Let Γ be a Fuchsian group.*

1. *If Γ is finite, it is a cyclic group generated by a rotation about a point by $2\pi/n$ radians, for some positive integer n.*
2. *If Γ is infinite but consists entirely of elliptic and parabolic elements, then it is infinite cyclic, is generated by a single parabolic element, and contains no elliptic elements.*
3. *If Γ contains a hyperbolic element γ that generates a subgroup of finite index, then there are two possibilities: either the group is infinite cyclic, generated by a hyperbolic element, or it has a subgroup of index 2, generated by a hyperbolic element.*

4. *In all other cases,* Γ *contains a subgroup that is isomorphic to the free group on two generators and consists entirely of hyperbolic elements. Such* Γ *are said to be "non-elementary".*

PROOF 1. Suppose Γ contains two elliptic elements, γ and δ, with distinct fixed points a and b. Then among the fixed points of the conjugates $\gamma^n\delta\gamma^{-n}$ and the fixed points of the conjugates $\delta^n\gamma\delta^{-n}$ there are two that are further apart than $d(a,b)$; see Figure 3.1.1.

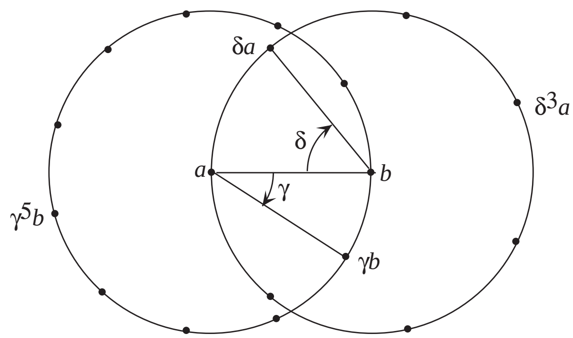

FIGURE 3.1.1 Above, γ is rotation clockwise around a and δ is rotation counterclockwise around b. In the cyclic groups generated by these rotations, there is always some element that rotates by at least $2\pi/3$, and if a and b are rotated by at least this amount, their images will be further apart than a and b.

Repeat the argument, using the conjugates of γ and δ having these fixed points, to find infinitely many fixed points of elliptics. This contradicts the claim that Γ is finite. Thus all elements of Γ have the same fixed point, and putting this fixed point at the origin in $\mathbf{D}$, we see that every element of Γ can be written $z \mapsto \lambda z$ with $|\lambda| = 1$. But the discrete subgroups of the unit circle are all finite cyclic groups.

2. By part 1, if Γ is an infinite discrete Fuchsian group made up of elliptic elements, then there must be angles 2α and 2β such that Γ contains rotations γ, δ by these angles with centers a and b arbitrarily far apart. Let m be the line joining a and b, let l_1 be a line through a making angle α with m, and let l_2 be a line through b making angle β with m, as shown in Figure 3.1.2. At both a and b there are two such lines; choose the appropriate ones so that $\gamma = R_m \circ R_{l_1}$ and $\delta = R_m \circ R_{l_2}$, where R_l denotes reflection in a line l. In any case,

$$\eta := \gamma^{-1} \circ \delta = R_{l_1} \circ R_{l_2} = R_{l_1} \circ R_m \circ R_m \circ R_{l_2} \qquad 3.1.1$$

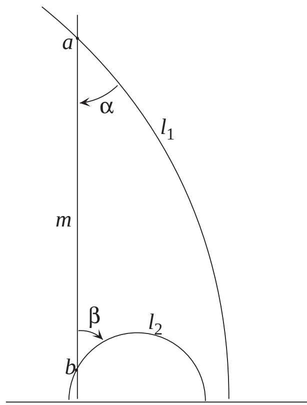

FIGURE 3.1.2. Given angles $\alpha, \beta > 0$, if two lines l_1, l_2 intersect a third line m under these angles at points a, b, then when a and b are sufficiently far apart, the lines l_1, l_2 are disjoint.

belongs to Γ. But if a and b are sufficiently far apart, l_1 and l_2 do not intersect, so η is not elliptic. This shows that an infinite Fuchsian group cannot be entirely made up of elliptic elements.

Suppose Γ contains a parabolic element γ and no hyperbolic elements. Use the **H** model of the hyperbolic plane. By conjugation and replacing γ by γ^{-1} if necessary, we may assume that $\gamma : z \mapsto z+1$. If another parabolic δ has a different fixed point, we may put that fixed point at 0, so that

$$\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \delta = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \tag{3.1.2}$$

for some $a \neq 0$. Exchanging δ and δ^{-1} if necessary, we may assume $a > 0$, and then $\gamma\delta$ has trace $2 + a$ and is hyperbolic.

Thus all the parabolics fix ∞ and all are translations by elements in some discrete subgroup of $\mathbb{R}$. But we know that such a subgroup is infinite cyclic, generated by some $t \in \mathbb{R}$. Any elliptics that Γ might contain must fix infinity, so there aren't any.

3. Suppose $\gamma \in \Gamma$ is hyperbolic and generates a subgroup of finite index; let its fixed points be a and b, which we may place at 0 and ∞. Hyperbolic elements with these fixed points are multiplication by positive reals, so $\langle \gamma \rangle$ is isomorphic to a discrete subgroup of $\mathbb{R}_+^*$ (the strictly positive reals), hence infinite cyclic, generated by some δ with $\delta^n = \gamma$ for some n.

Any element $\alpha \in \Gamma$ must preserve $\{a, b\}$: if $\alpha(\{a, b\}) = \{a', b'\}$, then a', b' are the fixed points of the subgroup $\alpha \circ \langle \delta \rangle \circ \alpha^{-1}$. If $\{a', b'\} \neq \{a, b\}$, then the orbit of $\{a', b'\}$ under $\langle \delta \rangle$ is infinite, giving infinitely many subgroups of Γ conjugate to $\langle \delta \rangle$, which is then not of finite index.

Thus there is a homomorphism $\Gamma \to \mathrm{Perm}\{a, b\}$; its kernel is $\langle \delta \rangle$, and if it is surjective, then the elements of Γ that exchange a and b are all conjugate, all elliptics of order 2.

4. Suppose Γ contains a hyperbolic element γ such that $\langle \gamma \rangle$ is not of finite index in Γ. We saw in part 3 that if all elements of Γ preserve the

set $\{a, b\}$ of fixed points of γ, then $\langle\gamma\rangle$ is of finite index in Γ. So there is an element $\delta \in \Gamma$ that does not preserve $\{a, b\}$, and $\gamma' := \delta \circ \gamma \circ \delta^{-1}$ is a hyperbolic element of Γ, such that γ' and γ are not powers of a single element, since they do not have the same fixed points in $\overline{\mathbb{R}}$.

In fact they have no common fixed point. If they did, put this common fixed point at infinity in the model $\mathbf{H}$, and the other fixed point of γ at 0. Then γ becomes the mapping $z \mapsto az$, and switching γ and γ^{-1} if necessary, we may assume $a > 1$. The element γ' is also affine, i.e., $\gamma'(z) = a'z + b'$ for some a', b' with $a' \neq 0$.

There can then be no translation in Γ: if $\delta := z \mapsto z + b'$, then the map $\gamma^{-n}\delta\gamma^n : z \mapsto z + b'/a^n$ converges to the identity, which contradicts the hypothesis that Γ is discrete. But

$$\left(\gamma^{-1} \circ (\gamma')^{-1} \circ \gamma \circ \gamma'\right)(z) = z + \frac{(a-1)b'}{aa'} \qquad 3.1.3$$

is a translation, contradicting the assumption that γ' and γ have a common fixed point.

Thus all the fixed points of γ and γ' are distinct. If the axes of γ and γ' intersect, then the axes of γ and $\gamma'' = \gamma'\gamma(\gamma')^{-1}$ do not intersect; in this case, rename γ'' to be γ'.

We now have two hyperbolic elements γ, γ' of Γ with disjoint axes. Consider the common perpendicular L to the axes, and powers γ^k, $(\gamma')^l$ such that the lines

$$\gamma^k(L), \quad \gamma^{-k}(L), \quad (\gamma')^l(L), \quad (\gamma')^{-l}(L) \qquad 3.1.4$$

are all disjoint; this is possible since these four lines are in arbitrarily small neighborhoods of the four distinct fixed points. Finally, the group Γ_1 generated by $\gamma^{2k}, (\gamma')^{2l}$ is a Schottky group (see Example 3.9.7); in particular, it is a free group on its two generators, and the quotient $\mathbf{D}/\Gamma_1$ is a sphere with three discs removed. $\square$

3.2 The classification of annuli

In this section we study cases 2 and 3 of Proposition 3.1.2, when the Fuchsian groups are torsion free. This is exactly equivalent to the study of Riemann surfaces homeomorphic to annuli.

A Riemann surface will be called an *annulus* if its fundamental group is isomorphic to $\mathbb{Z}$. We will see in a moment that this is equivalent to requiring that it be homeomorphic – in fact, analytically isomorphic – to some standard cylinder.

Proposition 3.2.1 *An annulus A is analytically isomorphic to either*

1. *the punctured plane* $\mathbb{C} - \{0\}$,
2. *the round annulus*

$$A_M := \Big\{ z \in \mathbb{C} \,|\, 1 < |z| < e^{2\pi M} \Big\}, \qquad 3.2.1$$

for exactly one value of $M \in (0, \infty)$, called the modulus of A, denoted $\operatorname{Mod}(A)$, *or*

3. *the punctured disc* $\mathbf{D}^* := \mathbf{D} - \{0\}$, *isomorphic to the exterior punctured disc*

$$A_\infty := \Big\{ z \in \mathbb{C} \,|\, 1 < |z| < \infty \Big\}. \qquad 3.2.2$$

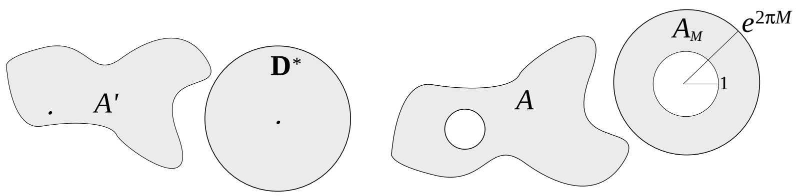

FIGURE 3.2.1 The annulus A' at far left is isomorphic to the punctured disc $\mathbf{D}^*$. The annulus A is isomorphic to the round annulus A_M. We do not show an annulus isomorphic to the punctured plane.

Remark 3.2.2 An annulus isomorphic to the punctured plane $\mathbb{C} - \{0\}$ is called *doubly infinite*. An annulus isomorphic to the punctured disc is called *singly infinite*. △

PROOF The universal covering space $\widetilde{A}$ must be isomorphic to either $\mathbb{C}$ or $\mathbf{D}$, by the uniformization theorem, Theorem 1.1.1. (It can't be compact, since the covering group is infinite.) By Theorem 1.8.2, the automorphisms of $\mathbb{C}$ are the mappings $z \mapsto az + b$, which always have a fixed point if $a \neq 1$. Thus if $\widetilde{A}$ is isomorphic to $\mathbb{C}$, the group of covering automorphisms is generated by a single translation, say $T_b : z \mapsto z + b$. The mapping $z \mapsto e^{2\pi i z/b}$ then induces an isomorphism $A \to \mathbb{C} - \{0\}$.

If $\widetilde{A}$ is isomorphic to $\mathbf{D}$, then the covering group is generated by a single automorphism α with no fixed point, which is either parabolic or hyperbolic (Proposition 2.1.14). If α is parabolic, then an isomorphism from $\mathbf{D}$ to $\mathbf{H}$ can be chosen so that α is conjugate to $z \mapsto z \pm 1$. As above, the map $z \mapsto e^{-2\pi i z}$ then induces an isomorphism $A \to A_\infty$.

Finally, if α is hyperbolic, then A is isomorphic to $\mathbf{B}/D(\alpha)\mathbb{Z}$, where D is the infimum defined in equation 2.1.13:

$$D(\alpha) := \inf_{z \in \mathbf{D}} d\big(z, \alpha(z)\big). \qquad 3.2.3$$