2

Plane hyperbolic geometry

In this chapter we will see that the unit disc $\mathbf{D}$ has a natural geometry, known as *plane hyperbolic geometry* or *plane Lobachevski geometry.* It is the local model for the hyperbolic geometry of Riemann surfaces, the subject of Chapter 3, so this chapter is a prerequisite for the next.

Plane hyperbolic geometry is essentially an elementary subject, similar to Euclidean geometry, and even more similar to spherical trigonometry. Sections 2.1 and 2.4 could be taught in an undergraduate geometry course, and often are. Section 2.2 discusses curvature; Section 2.3 shows that canoeing in the hyperbolic plane would be very different from canoeing in the Euclidean plane: in the hyperbolic plane, if you deviate only slightly from the straight line, your canoe will not go around in circles.

2.1 The hyperbolic metric

The disc has a natural metric, invariant under all automorphisms: the hyperbolic metric. In our usage, the hyperbolic metric will be understood to be an *infinitesimal metric*, i.e., a way to measure tangent vectors, given by a norm on each tangent space. Such an infinitesimal metric induces a metric in the ordinary sense, via lengths of curves; we discuss this below.

In general, an infinitesimal metric on an open subset $U \subset \mathbb{R}^2$ is written as a positive definite (real) quadratic form

$$E(x,y)\,dx^2 + 2F(x,y)\,dx\,dy + G(x,y)\,dy^2. \qquad 2.1.1$$

(The letters E, F, G for the coefficients are traditional and go back to Gauss.) This infinitesimal metric assigns the length

$$\sqrt{E(x,y)a^2 + 2F(x,y)ab + G(x,y)b^2} \qquad 2.1.2$$

to the vector $(a,b) \in T_{(x,y)}U$.

The hyperbolic metric and most of the other metrics relevant to us will be *conformal*: they are metrics on Riemann surfaces, and if U is a Riemann surface, then the entries of the metric in an analytic coordinate $z = x + iy$ satisfy $E = G$ and $F = 0$. Setting $|dz|^2 := dx^2 + dy^2$, we can thus write conformal metrics as

$$E(x,y)(dx^2 + dy^2) = \big(\varphi(z)\big)^2 |dz|^2, \qquad 2.1.3$$

with φ a positive real-valued function on $U \subset \mathbb{C}$. We denote by $\varphi(z)|\xi|$ the length assigned to the tangent vector $\xi \in T_zU$ by this metric. (We will often call $\varphi(z)|dz|$ the metric; in this way of thinking, the metric returns a length, not length squared.) Conformal metrics interact well with multiplication by complex numbers: if α is a complex number, then

$$\big(\varphi(z)|dz|\big)(\alpha\xi) = \varphi(z)\Big(|dz|(\alpha\xi)\Big) = |\alpha|\big(\varphi(z)|dz|\big)(\xi) = |\alpha|\varphi(z)\,|\xi|. \quad 2.1.4$$

Exercise 2.1.1 Show that there is no metric on $\mathbb{C}$ or on $\mathbb{P}^1$ that is invariant under all analytic automorphisms. ♢

Proposition and Definition 2.1.2 (Hyperbolic metric on the disc) *All analytic automorphisms of* $\mathbf{D}$ *are isometries for the (infinitesimal) hyperbolic metric*

$$\rho_{\mathbf{D}} := \frac{2|dz|}{1-|z|^2}. \quad 2.1.5$$

All invariant metrics are multiples of the hyperbolic metric.

The hyperbolic metric is also called the *Poincaré metric*.

Figure 2.1.1, right, illustrates the hyperbolic metric on the unit disc.

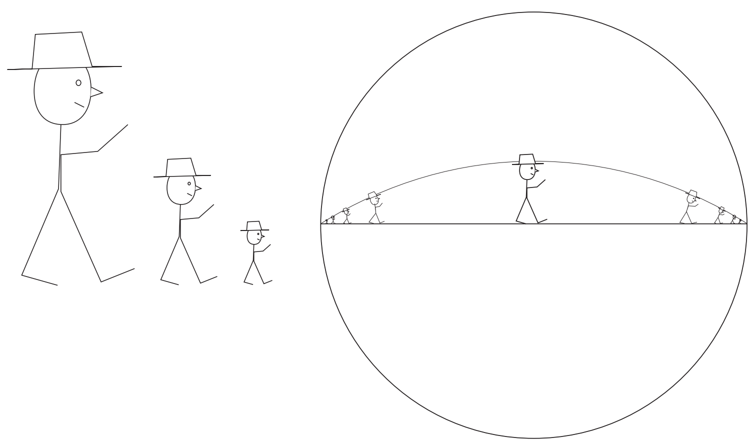

FIGURE 2.1.1 LEFT: Stickmen of different sizes. RIGHT: Measured with the hyperbolic metric, these stickmen in $\mathbf{D}$ are all the same size and are spaced an equal distance apart. The men are walking on the part of the real axis in the unit disc, which is a geodesic; their hats are on a curve at constant distance from this geodesic. This curve is a circle of hyperbolic geometry, i.e., a curve of constant geodesic curvature (see Definition 2.3.3), but it is not itself a geodesic. The points where the curve of feet and the curve of hats appear to meet are points at infinity.

PROOF We can construct an invariant metric $\varphi(z)|dz|$ as follows. First, choose $\varphi(0) > 0$. If $z \in \mathbf{D}$, then the automorphism

$$w \mapsto \frac{w - z}{1 - \bar{z}w} \tag{2.1.6}$$

maps z to 0, and its derivative maps the tangent vector $\xi \in T_z\mathbf{D}$ to the tangent vector $\xi/(1 - |z|^2) \in T_0\mathbf{D}$, so

$$\underbrace{\varphi(z)|dz|(\xi)}_{\text{metric applied to } \xi} = \underbrace{\varphi(0)\frac{|\xi|}{1 - |z|^2}}_{\text{length of the image of } \xi \text{ under the automorphism}}. \tag{2.1.7}$$

This is well defined, because any other automorphism mapping z to 0 must differ from the one given by 2.1.6 by a rotation around 0, which will preserve $|\xi|$. This shows that equation 2.1.7 does define an invariant metric, that all invariant metrics are of this form, and that all are conformal. Remark 2.1.10 discusses why $\varphi(0)$ is chosen to be 2 in equation 2.1.5, and not 1, as you might expect. □

It is often convenient to have other models of the hyperbolic plane. By the uniformization theorem, any simply connected noncompact Riemann surface other than $\mathbb{C}$ is a model of the hyperbolic plane, and if you can write down a conformal mapping explicitly, you can find the hyperbolic metric for that model explicitly. The following models are especially useful:

1. the band $\mathbf{B} := \{ z \in \mathbb{C} \mid |\operatorname{Im} z| < \pi/2 \}$ with the hyperbolic metric $|dz|/\cos \operatorname{Im} z$, shown in Figures 2.1.2 and 2.1.3.
2. the upper halfplane $\mathbf{H}$ with the hyperbolic metric $|dz|/\operatorname{Im} z$, shown in Figure 2.1.4. (When we need to consider the lower halfplane, we will denote it $\mathbf{H}^*$.) Note that the real axis is not part of $\mathbf{H}$.

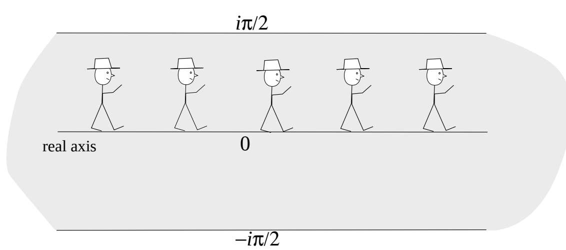

FIGURE 2.1.2 Stickmen walking on the real axis in the band model $\mathbf{B}$ of the hyperbolic plane. In this model, on the real axis, Euclidean and hyperbolic lengths coincide. We saw in Figure 2.1.1 that this is not true of the disc model.

Note that there is a natural unit of length in the hyperbolic plane: the one that assigns curvature -1 to the plane; see Remark 2.1.10. So we do not

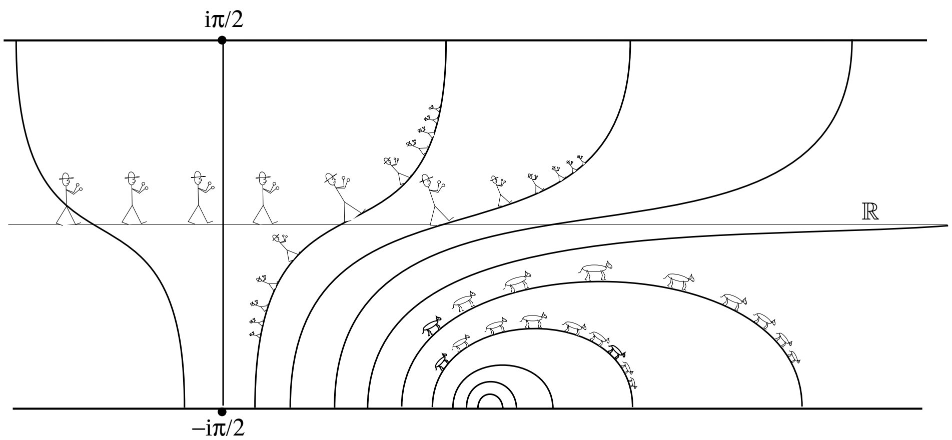

FIGURE 2.1.3 Here we see many geodesics in the band model **B**: some that do not intersect $\mathbb{R}$ (with dogs walking on some of them), one that is asymptotic to it, and some that intersect it, with stickmen walking on some of them. The dogs (on the scale of the stickmen) are roughly the size of Great Danes. The stickmen are all the same size, and are regularly spaced about .5 apart.

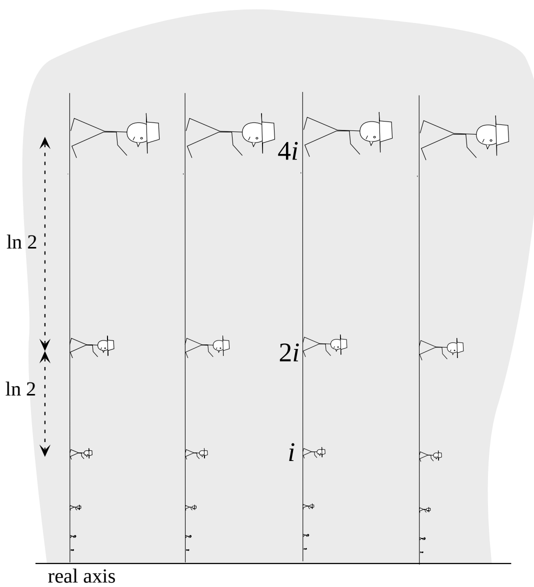

FIGURE 2.1.4. Stickmen in the upper halfplane model **H** of the hyperbolic plane. Every vertical line is a geodesic. The stickmen on one vertical line are all the same size and are equally spaced (two men next to each other are $\ln 2$ apart). Although the vertical lines look parallel, the lines are asymptotic and the distance between two adjacent lines is 0. Stickmen at height $2i$ are twice as far apart as those at height $4i$.

need to specify the unit of length. This is analogous to deciding that a sphere has radius 1 because that is the radius that gives curvature 1 to the sphere.

Exercise 2.1.3 Show that **H** and **B** are isometric to **D**. $\diamondsuit$

Exercise 2.1.4 1. Show that the complex analytic automorphisms of **H** are the maps $z \mapsto \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} > 0$.

2. Show that this identifies $\operatorname{Aut} \mathbf{H}$ with $\mathrm{PSL}_2 \mathbb{R} := \mathrm{SL}_2 \mathbb{R} / \pm I$.

3. Show that $\mathrm{PSL}_2 \mathbb{R}$ is precisely the set of orientation-preserving isometries of **H** for the hyperbolic metric. ♢

Exercise 2.1.5 1. Find the hyperbolic metric of $\mathbb{C} - [0, \infty)$.

2. Find the hyperbolic metric of $\mathbf{D} - [0, 1)$. ♢

There is a nice restatement of Schwarz's lemma in terms of the hyperbolic metric:

Proposition 2.1.6 (Schwarz-Pick theorem)

1. *All analytic maps $f \colon \mathbf{D} \to \mathbf{D}$ are weakly contracting for the hyperbolic metric.*
2. *If such an f is an isometry at a single point, it is an automorphism.*

PROOF 1. Choose $z \in \mathbf{D}$ and automorphisms $\alpha, \beta \colon \mathbf{D} \to \mathbf{D}$ such that $\alpha(0) = z$ and $\beta(f(z)) = 0$. Then $\beta \circ f \circ \alpha$ maps **D** to **D** and takes 0 to 0. The standard form of Schwarz's lemma now says that this mapping is weakly contracting, i.e., $|(\beta \circ f \circ \alpha)'(0)| \le 1$. Part 1 follows from the fact that α and β are isometries.

2. If f is an isometry at z, then the derivative of $\beta \circ f \circ \alpha$ at 0 has absolute value 1, so $\beta \circ f \circ \alpha$ is a rotation (again, by the standard Schwarz's lemma), hence an automorphism. Hence so is f. □

The models **D**, **H**, and **B** are summarized in Table 2.1.6.

Geodesics

With an infinitesimal metric, we can measure lengths of rectifiable curves. This allows us to define the distance between two points to be the infimum of the lengths of the curves joining them. For example, the *hyperbolic distance* is the distance using the hyperbolic metric. Curves that minimize length are the *geodesics* of the geometry. It is easy to say exactly what they are for the hyperbolic plane, especially in the model **H**; see Figure 2.1.5 and Proposition 2.1.7.

Proposition 2.1.7 (Geodesics in H) *Given any two points a, b in the upper halfplane **H**, there exists a unique semicircle perpendicular to the real axis and passing by a and b. (If $\mathrm{Re}\,(a) = \mathrm{Re}\,(b)$, the semicircle degenerates to the vertical line through a and b.) The arc of this semicircle that joins a to b is the unique geodesic arc joining these points.*