1
The uniformization theorem

Thurston's basic insight in all four of the theorems discussed in this book is that either the topology of the problem induces an appropriate geometry or there is an understandable obstruction. The ancestor of such a statement is the uniformization theorem, which asserts that every simply connected Riemann surface carries a natural geometry, either spherical (the Riemann sphere), Euclidean (the complex plane), or hyperbolic (the unit disc).

It follows from the uniformization theorem that *all* Riemann surfaces have a natural geometry – spherical, Euclidean, or hyperbolic. We will discuss this in Section 1.8 (Theorem 1.8.8).

Most Riemann surfaces are hyperbolic, and this hyperbolic structure is the backbone of the entire book.

1.1 Two statements of the theorem

A Riemann surface is a complex analytic manifold of dimension 1.

Theorem 1.1.1 (The uniformization theorem) *A simply connected Riemann surface is isomorphic to either the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$, or the open unit disc $\mathbf{D} \subset \mathbb{C}$.*

Observe that the surfaces are indeed distinct: the Riemann sphere $\mathbb{P}^1$ is compact, whereas $\mathbb{C}$ and $\mathbf{D}$ are not; there are nonconstant bounded analytic functions on $\mathbf{D}$ but not on $\mathbb{C}$, by Liouville's theorem.

We will actually prove the slightly different Theorem 1.1.2, using cohomology rather than the fundamental group.

Theorem 1.1.2 *If a Riemann surface X is connected and noncompact and its cohomology satisfies $H^1(X,\mathbb{R}) = 0$, then it is isomorphic either to $\mathbb{C}$ or to $\mathbf{D}$.*

Sections 1.2–1.7 are devoted to proving this theorem.

Theorem 1.1.2 appears to be both stronger and weaker than Theorem 1.1.1. It appears to be stronger because the hypothesis concerns cohomology rather than the fundamental group. Recall that for any connected topological space, $H^1(X,\mathbb{R}) = \mathrm{Hom}\big(\pi_1(X,x),\mathbb{R}\big)$, so if X is simply connected, then $H^1(X,\mathbb{R}) = 0$, but the converse is false in general. Thus one consequence of the theorem is that if the cohomology of a Riemann surface is trivial, then so is the fundamental group.

It appears to be weaker because it requires that X be noncompact. However, Theorem 1.1.3 shows that the uniformization theorem for compact surfaces follows from Theorem 1.1.2, which therefore really is stronger than Theorem 1.1.1.[1]

Theorem 1.1.3 *Let X be a connected compact Riemann surface satisfying $H^1(X, \mathbb{R}) = 0$. Then X is isomorphic to $\mathbb{P}^1$.*

Proof of Theorem 1.1.3 from Theorem 1.1.2 It is enough to prove that if $x \in X$ is a point, then $X' := X - \{x\}$ is isomorphic to $\mathbb{C}$; by the removable singularity theorem, that implies that X is isomorphic to $\mathbb{P}^1$. First, let us see that $H^1(X', \mathbb{R}) = 0$.

Lemma 1.1.4 *If a compact connected surface X satisfies $H^1(X, \mathbb{R}) = 0$, then for any $x \in X$, the surface $X' := X - \{x\}$ satisfies $H^1(X', \mathbb{R}) = 0$.*

Proof Let U be a neighborhood of x homeomorphic to a disc; the Mayer-Vietoris exact sequence of $(X; X', U)$ gives

$$\ldots \to \underbrace{H^1(X,\mathbb{R})}_{0 \text{ by hyp.}} \to H^1(X',\mathbb{R}) \oplus H^1(U,\mathbb{R}) \to \underbrace{H^1(X' \cap U,\mathbb{R})}_{\cong\mathbb{R}} \to \underbrace{H^2(X,\mathbb{R})}_{\cong\mathbb{R}} \to 0.$$

The sequence ends in 0 because neither X' nor U has a compact component. The first term vanishes by hypothesis; the third and fourth are isomorphic to $\mathbb{R}$, since $X' \cap U$ is homeomorphic to a disc with the origin removed, and since X is a compact, orientable, connected surface. The map connecting them is an isomorphism, since it is linear and surjective; this shows that the second term vanishes also. □

Suppose X' is not isomorphic to $\mathbb{C}$; then by Theorem 1.1.2 it must be isomorphic to $\mathbf{D}$. But X is the one-point compactification of X', so it must be the one-point compactification of $\mathbf{D}$. This is impossible, by the following lemma, so X' is isomorphic to $\mathbb{C}$.

Lemma 1.1.5 *The one-point compactification $\overline{\mathbf{D}} := \mathbf{D} \cup \{\infty\}$ of $\mathbf{D}$ does not carry a Riemann surface structure coinciding with the standard structure of $\mathbf{D}$.*

[1] The distinction between homology and the fundamental group is not a triviality. Poincaré's first version of the Poincaré conjecture was that a compact 3-dimensional manifold M such that $H_1(M, \mathbb{Z}) = 0$ is homeomorphic to the 3-sphere S^3. Within a year he discovered the "Poincaré dodecahedral space", a compact 3-dimensional manifold whose homology vanishes but whose fundamental group has 120 elements; he recast his conjecture to say that a compact simply connected 3-dimensional manifold is homeomorphic to S^3.

PROOF The analytic (identity) function z on $\mathbf{D} = \overline{\mathbf{D}} - \{\infty\}$ is bounded in a neighborhood of ∞, so if $\overline{\mathbf{D}}$ carries a Riemann surface structure, then z should extend as an analytic function on $\overline{\mathbf{D}}$ by the removable singularity theorem. But it doesn't even extend as a continuous function. □

Thus Theorem 1.1.3 follows from Theorem 1.1.2, and the two together give Theorem 1.1.1.

1.2 Subharmonic and harmonic functions

In this section we prove that certain harmonic extensions exist (Proposition 1.2.4). We will need this result in two places: first, to show that Riemann surfaces admit partitions of unity (Rado's theorem, Theorem 1.3.3), which will be essential for our construction of an exhaustion of a simply connected Riemann surface by compact simply connected subsets; second, when we construct Green's functions on these pieces (Proposition 1.5.1).

The value of a function at the center of a circle is either $\geq$, $\leq$, or $=$ to the average value of the function on the circle. We assign special names to functions where this happens for all points and all circles.

Definitions 1.2.1 (Harmonic, subharmonic, superharmonic) Let X be a Riemann surface. A continuous function $f : X \to \mathbb{R}$ is *harmonic* if for every chart $\varphi : U \to X$ with $U \subset \mathbb{C}$ open and every circle $|\zeta - \zeta_0| = r$ in U, the difference

$$\left(\frac{1}{2\pi} \int_0^{2\pi} f\Big(\varphi\left(\zeta_0 + re^{i\theta}\right)\Big)\, d\theta \right) - f\left(\varphi(\zeta_0)\right) \qquad 1.2.1$$

is zero. If the difference is nonnegative, then f is *subharmonic*. If it is nonpositive, then f is *superharmonic*.

Subharmonic and superharmonic functions are not very interesting in their own right, but they are easy to construct, largely because we can take sups of subharmonic functions and infs of superharmonic functions. As we will see in Proposition 1.2.3, this will allow us to construct harmonic functions, which are the objects of interest.

Subharmonic functions satisfy the *maximum principle*: if the domain of a subharmonic function f is connected and f has a local maximum, then f is constant.

Recall that any continuous function on the boundary of a closed disc in $\mathbb{C}$ extends to a harmonic function in the interior of the disc by the Poisson integral formula.

Definition 1.2.2 (Bounded Perron family) Let M be a real number. A set of subharmonic functions $\mathcal{F}$ on a Riemann surface X is called a *Perron family bounded by M* if it satisfies the following requirements:

1. If $f \in \mathcal{F}$, then $|f| \le M$.
2. If $f_1, f_2 \in \mathcal{F}$, then $\sup(f_1, f_2) \in \mathcal{F}$.
3. Let $f \in \mathcal{F}$ be a function and let D be a disc in the image of a chart of X. If f_1 is the continuous function that is f outside D and harmonic in D, then $f_1 \in \mathcal{F}$.

The next statement constructs harmonic functions from subharmonic functions; it is the main result we will need about subharmonic functions.

Proposition 1.2.3 (Perron's theorem) *If $\mathcal{F}$ is a nonempty bounded Perron family on a Riemann surface X, then $F := \sup \mathcal{F}$ is harmonic.*

Proof Choose $z_0 \in X$ and a neighborhood U of z_0 on which there exists a chart $\zeta : U \to \mathbb{C}$ such that $\overline{\Delta} := \{|\zeta| \le 1\}$ is a compact disc in U. There exists a sequence $f_n \in \mathcal{F}$ such that $\sup f_n(z_0) = F(z_0)$. By replacing f_n by $\sup(f_1, \dots, f_n)$, we may assume that $f_n(x) \le f_{n+1}(x)$ for every $x \in X$ and every n, i.e., that the sequence is monotone increasing at every point.

Let $\tilde{f}_n$ be the continuous function equal to f_n outside Δ and harmonic in Δ. Since $\mathcal{F}$ is Perron, we have $\tilde{f}_n \in \mathcal{F}$. Since $\tilde{f}_n \ge f_n$, we have $\sup \tilde{f}_n(z_0) = F(z_0)$. By Harnack's principle, $\sup \tilde{f}_n$ is harmonic on Δ.

Thus if we can prove that $F = \sup \tilde{f}_n$ in Δ, we will be done. Let z_1 be a point in Δ, and construct as above an increasing sequence g_n such that $\sup g_n(z_1) = F(z_1)$. Set $h_n := \sup(f_n, g_n)$ and define $\tilde{h}_n$ to be the continuous function equal to h_n outside Δ and harmonic in Δ. Then $\sup \tilde{h}_n$ is a harmonic function on Δ. The harmonic function $\sup \tilde{h}_n - \sup \tilde{f}_n$ is ≥ 0 in Δ and achieves its minimum 0 at z_0, so it is identically 0. Thus

$$F(z_1) = \sup \tilde{h}_n(z_1) = \sup \tilde{f}_n(z_1). \qquad \square \qquad 1.2.2$$

We will need the following proposition in order to find a Green's function, but it is of great interest in its own right. It is due to Oskar Perron (1880–1975). Solving Laplace's equation with given boundary conditions goes under the name of *Dirichlet's problem*, and is one of the fundamental problems of partial differential equations. Compared to other solutions, Perron's stands out for its simplicity and the weakness of the hypotheses. In particular, the proposition does not require that X be second countable.

Proposition 1.2.4 (Existence of harmonic functions) *Let $m \leq M$ be two real numbers and let X be a subsurface of a Riemann surface Y with nonempty smooth boundary ∂X. Let $f: \partial X \to [m, M]$ be a bounded continuous function. Then there exists a continuous function $\tilde{f}: X \to [m, M]$ that is harmonic on the interior of X and equals f on the boundary of X.*

Proof Consider the family $\mathcal{F}$ of continuous functions $g: X \to [m, M]$ that are subharmonic on the interior and such that $g \leq f$ on ∂X. Since the constant function m belongs to $\mathcal{F}$, the family is nonempty. It is a bounded Perron family, therefore has a supremum $\tilde{f}$ that is harmonic on the interior of X. We need to see that $\tilde{f}$ is continuous on X and agrees on the boundary with f.

Let x be a point of ∂X, let U be a neighborhood of x in Y, and let $\zeta: U \to \mathbb{C}$ be a local coordinate[2] on Y with $\zeta(x) = 0$. Let $(x_n)_{n \geq 0}$ be a sequence in $U - X$, tending to x on the line orthogonal at x to ∂X, in the coordinate ζ. Then for any $\epsilon > 0$, the function

$$h_{n,\epsilon}(z) := \sup\left(m,\ \ln\left|\frac{\zeta(x_n)}{\zeta(z) - \zeta(x_n)}\right| + f(x) - \epsilon\right) \qquad 1.2.3$$

belongs to $\mathcal{F}$ for n sufficiently large. This is true because the function

$$\ln\left|\frac{\zeta(x_n)}{\zeta(z) - \zeta(x_n)}\right| \qquad 1.2.4$$

tends uniformly to $-\infty$ on the complement of any compact neighborhood of x in U as $n \to \infty$, hence off such a neighborhood, the supremum is realized by m. In particular there is no discontinuity on the boundary of U.[3]

Similarly, the function

$$k_{n,\epsilon}(z) := \inf\left(M,\ \ln\left|\frac{\zeta(z) - \zeta(x_n)}{\zeta(x_n)}\right| + f(x) + \epsilon\right) \qquad 1.2.5$$

is for n sufficiently large a superharmonic function greater than f on ∂X. Therefore any $g \in \mathcal{F}$ satisfies $g < k_{n,\epsilon}$ for n sufficiently large; see Figure 1.2.1. Using $h_{n,\epsilon}$ we see that $\liminf_{z \to x} \tilde{f} \geq f(x)$, and using $k_{n,\epsilon}$ we see that $\limsup_{z \to x} \tilde{f} \leq f(x)$. □

[2]See "local coordinate" and "chart" in the glossary.

[3]Recall that if f is an analytic function, then $\ln|f|$ is harmonic where $f \neq 0$, since $\ln|f|$ is locally the real part of $\ln f$.

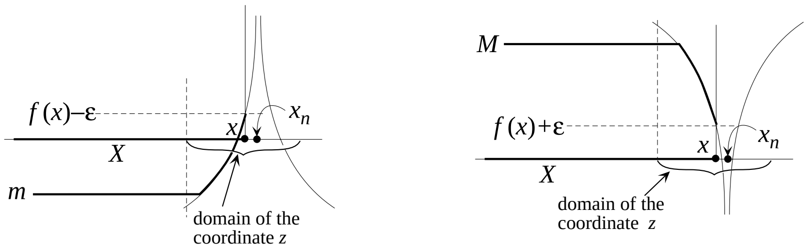

FIGURE 1.2.1 LEFT: The subharmonic function $h_{n,\epsilon}$. RIGHT: The superharmonic function $k_{n,\epsilon}$. The key point is that the constants m and M realize the sup and inf respectively off a compact subset of the domain of the local coordinate z.

Example 1.2.5 (Small boundaries are bad) In order to have $\tilde{f}|_{\partial X} = f$, we must know something about the boundary (and in Proposition 1.2.4, we do: the topological boundary is the boundary of a manifold with boundary). For instance, the family $\mathcal{F}$ of continuous functions f on $\overline{\mathbf{D}}$ that are subharmonic on $\mathbf{D} - \{0\}$ and such that $-1 \le f \le 0$ and $f(0) = -1$ is clearly a Perron family. But the function

$$f_\epsilon := \sup(\epsilon \ln |z|, -1) \qquad 1.2.6$$

belongs to $\mathcal{F}$ for all $\epsilon > 0$, and so $\sup \mathcal{F}(z) = 0$ for all $z \in \mathbf{D} - \{0\}$. Thus the boundary value -1 is not achieved. △

1.3 RADO'S THEOREM

In this section we show that every connected Riemann surface is second countable, i.e., there is a countable basis for the topology.[4] We need this for two reasons. For one thing, otherwise the uniformization theorem would obviously be wrong as stated. In addition, we will need partitions of unity and in Appendix A1 we show that every second countable finite-dimensional manifold admits a partition of unity subordinate to any cover.

As it is rather hard to imagine any surface that is not second countable, we begin with an example.

Example 1.3.1 (A horrible surface) Consider the disjoint union

$$X := \mathbf{H} \sqcup \bigsqcup_{x \in \mathbb{R}} \overline{\mathbf{H}}_x, \qquad 1.3.1$$

where $\mathbf{H}$ is the upper halfplane $\mathbf{H} := \{\, z \in \mathbb{C} \mid \operatorname{Im} z > 0 \,\}$, and $\overline{\mathbf{H}}_x$ is a copy of the closed lower halfplane, as shown in Figure 1.3.1.

[4]For a discussion of the relationship between second countable, σ-compact, and partitions of unity, see Appendix A1.