A1
Partitions of unity

In this appendix we prove that a second-countable finite-dimensional manifold has a partition of unity subordinate to any cover. This material, although easy, is often omitted in courses on manifolds.

Definition A1.1 (Second countable) A topological space X is *second countable* if there is a countable basis for the topology.

In other words, X is second countable if there is a countable collection of open sets $(U_i)_{i \in I}$ such that every open set U is a union

$$U := \cup_{j \in J} U_j \qquad A1.1$$

for some subset $J \subset I$. A standard example of a second-countable space is $\mathbb{R}^n$; we can take the U_i to be the balls with rational radii centered at points with rational coordinates. Separable Banach spaces are also second countable.

A standard counterexample is the nonseparable Banach space l^∞; the uncountably many unit balls centered at the vectors where all the entries are ± 1 are disjoint. The horrible surface of Example 1.3.1 is a much more relevant counterexample.

Definition A1.2 (σ-compact) A topological space X is *σ-compact* if it is Hausdorff and is a countable union of compact sets.

Proposition A1.3 *A locally compact Hausdorff space X that is second countable is σ-compact. In particular, every second-countable finite-dimensional manifold is σ-compact.*

Proof For each point $x \in X$, choose a neighborhood V_x with compact closure. If $\mathcal{U}$ is a countable basis for the topology, it is easy to see that those $U \in \mathcal{U}$ that are contained in some V_x are still a basis, and that $\overline{U}$ is compact for all such U. These sets $\overline{U}$ are a countable collection of compact sets whose union is X. $\square$

Exercise A1.4 shows that a topological space that is not locally compact can perfectly well be second countable without being σ-compact.

Exercise A1.4 Show that a Hilbert space with a countably infinite basis is second countable but not σ-compact. Hint: A compact subset is nowhere dense, so you can apply the Baire category theorem. $\diamondsuit$

Definition A1.5 (Locally finite cover) An open cover $\mathcal{U} := (U_i)_{i \in I}$ of a topological space X is *locally finite* if every point $x \in X$ has a neighborhood V that intersects only finitely many of the U_i.

The first step in constructing partitions of unity subordinate to an open cover is to show that the cover has a locally finite refinement. A slight modification of the proof of Proposition A1.6 shows that this is true for any σ-compact space. We will show a slightly stronger statement, but only for finite-dimensional manifolds; it will simplify the construction of partitions of unity.

Proposition A1.6 *Let $B_r^n \subset \mathbb{R}^n$ be the ball of radius r, and let X be a second countable, n-dimensional manifold. Then any open cover of X admits a countable, locally finite refinement $\mathcal{U}$ by open subsets U that admit surjective coordinate maps $\varphi_U : U \to B_2^n$ such that the $\varphi_U^{-1}(B_1^n)$ still cover X.*

Proof Since X is second countable, it has at most countably many components, and we may assume X to be connected. Choose a countable cover $\mathcal{V} := \{V_0, V_1, \dots\}$ of X by open sets with compact closures, indexed by the positive integers. Define by induction compact sets

$$A_0 \subset A_1 \subset \dots \qquad \text{A1.2}$$

as follows: Set $A_0 := \overline{V}_0$, and suppose $A_0, \dots, A_i$ have been defined. The V_j form an open cover of A_i, so there is a smallest J_i such that

$$A_i \subset \bigcup_{j \le J_i} U_j. \qquad \text{A1.3}$$

Set $A_{i+1} := \cup_{j \le J_i} \overline{V}_j$. If the J_i eventually stabilize, i.e., if $J_i = J_{i+1} = \dots$, then A_i is closed (in fact, compact) and open in X, hence $A_i = X$, since X is connected. Otherwise, $J_i \to \infty$ and we also have $\cup_i A_i = \cup_i \overline{V}_i = X$. In both cases, $\cup_i A_i = X$. For convenience, set $A_i := \emptyset$ if $i < 0$.

Let $\mathcal{W}$ be an open cover of X. Intersect all open sets $W \in \mathcal{W}$ with all $\mathring{A}_{i+2} - A_i$, to construct a new open cover $\mathcal{W}'$ that refines $\mathcal{W}$. For each $x \in X$, find a coordinate neighborhood U_x contained in some element of $\mathcal{W}'$, together with a surjective local coordinate $\varphi_x : U_x \to B_2^n$. Define $U_x' := \varphi_x^{-1}(B_1^n)$.

For each i, choose a cover of the compact set $A_i - \mathring{A}_{i-1}$ by finitely many of the $U_x', x \in Z_i$, where $Z_i \subset A_i - \mathring{A}_{i-1}$ is a finite set. For convenience, set

$Z_i := \emptyset$ for $i < 0$. The set $Z := \cup_i Z_i$ is countable; consider the cover $\mathcal{U}$ by all $U_x, x \in Z$. This cover is a refinement of $\mathcal{W}$, and it is locally finite: the open sets $\overset{\circ}{A}_{i+1} - A_{i-1}$ form an open cover of X, and each one can intersect only the finitely many

$$U_x, \text{ for } x \in \bigcup_{j=i-2}^{i+2} Z_j. \qquad A1.4$$

Clearly our open sets are coordinate neighborhoods, as desired. □

Construction of partitions of unity

It is now easy to construct partitions of unity subordinate to any cover of a second-countable n-dimensional manifold X.

Theorem A1.7 (Partitions of unity) *Let X be a second countable finite-dimensional topological manifold, and let $\mathcal{W}$ be an open cover of X. Then there exists a locally finite refinement $\mathcal{U}$ of $\mathcal{W}$, and a continuous partition of unity subordinate to $\mathcal{U}$. Moreover, if X is a C^∞ manifold, the partition of unity can be chosen C^∞.*

PROOF Find a locally finite cover $\mathcal{U} := (U_x)_{x \in Z}$, as in the proof of Proposition A1.6, together with coordinate maps φ_x, which can be chosen C^∞ if X is C^∞.

Let $h \geq 0$ be a C^∞ function on $\mathbb{R}^n$ with support in B_2^n and strictly positive on B_1^n. Define h_x, for $x \in Z$, by $h_x := h \circ \varphi_x$; since h_x has compact support, it is the restriction of a continuous function on X (C^∞ if X is C^∞) with support in U_x, which we will still denote by h_x. Now the function $g := \sum_{x \in Z} h_x$ is a strictly positive continuous function on X; the sum exists and is continuous (C^∞ if X is C^∞) because the cover $\mathcal{U}$ is locally finite. The functions g_x, $x \in Z$, defined by $g_x := h_x/g$ form the desired continuous partition of unity, C^∞ if X is C^∞. □