Partitions of unity

In this appendix we prove that a second-countable finite-dimensional manifold has a partition of unity subordinate to any cover. This material, although easy, is often omitted in courses on manifolds.

**Definition A1.1 (Second countable)** A topological space $X$ is **second countable** if there is a countable basis for the topology.

In other words, $X$ is second countable if there is a countable collection of open sets $(U_i)_{i \in I}$ such that every open set $U$ is a union

$$U := \bigcup_{j \in J} U_j$$

for some subset $J \subset I$. A standard example of a second-countable space is $\mathbb{R}^n$; we can take the $U_i$ to be the balls with rational radii centered at points with rational coordinates. Separable Banach spaces are also second countable.

A standard counterexample is the nonseparable Banach space $l^\infty$; the uncountably many unit balls centered at the vectors where all the entries are $\pm 1$ are disjoint. The horrible surface of Example 1.3.1 is a much more relevant counterexample.

**Definition A1.2 (σ-compact)** A topological space $X$ is **σ-compact** if it is Hausdorff and is a countable union of compact sets.

**Proposition A1.3** A locally compact Hausdorff space $X$ that is second countable is σ-compact. In particular, every second-countable finite-dimensional manifold is σ-compact.

**Proof** For each point $x \in X$, choose a neighborhood $V_x$ with compact closure. If $U$ is a countable basis for the topology, it is easy to see that those $U \in U$ that are contained in some $V_x$ that are still a basis, and that $\overline{U}$ is compact for all such $U$. These sets $\overline{U}$ are a countable collection of compact sets whose union is $X$. □

Exercise A1.4 shows that a topological space that is not locally compact can perfectly well be second countable without being σ-compact.
Exercise A1.4 Show that a Hilbert space with a countably infinite basis is second countable but not $\sigma$-compact. Hint: A compact subset is nowhere dense, so you can apply the Baire category theorem.

Definition A1.5 (Locally finite cover) An open cover $\mathcal{U} := (U_i)_{i \in I}$ of a topological space $X$ is locally finite if every point $x \in X$ has a neighborhood $V$ that intersects only finitely many of the $U_i$.

The first step in constructing partitions of unity subordinate to an open cover is to show that the cover has a locally finite refinement. A slight modification of the proof of Proposition A1.6 shows that this is true for any $\sigma$-compact space. We will show a slightly stronger statement, but only for finite-dimensional manifolds; it will simplify the construction of partitions of unity.

Proposition A1.6 Let $B^n_r \subset \mathbb{R}^n$ be the ball of radius $r$, and let $X$ be a second countable, $n$-dimensional manifold. Then any open cover of $X$ admits a countable, locally finite refinement $\mathcal{U}$ by open subsets $U$ that admit surjective coordinate maps $\varphi_U : U \to B^n_2$ such that the $\varphi_U^{-1}(B^n_1)$ still cover $X$.

Proof Since $X$ is second countable, it has at most countably many components, and we may assume $X$ to be connected. Choose a countable cover $\mathcal{V} := \{V_0, V_1, \ldots \}$ of $X$ by open sets with compact closures, indexed by the positive integers. Define by induction compact sets

$$ A_0 \subset A_1 \subset \ldots \quad A1.2 $$

as follows: Set $A_0 := \overline{V}_0$, and suppose $A_0, \ldots, A_i$ have been defined. The $V_j$ form an open cover of $A_i$, so there is a smallest $J_i$ such that

$$ A_i \subset \bigcup_{j \leq J_i} U_j. \quad A1.3 $$

Set $A_{i+1} := \bigcup_{j \leq J_i} \overline{V}_j$. If the $J_i$ eventually stabilize, i.e., if $J_i = J_{i+1} = \ldots$, then $A_i$ is closed (in fact, compact) and open in $X$, hence $A_i = X$, since $X$ is connected. Otherwise, $J_i \to \infty$ and we also have $\cup_i A_i = \cup_i \overline{V}_i = X$. In both cases, $\cup_i A_i = X$. For convenience, set $A_i := \emptyset$ if $i < 0$.

Let $\mathcal{W}$ be an open cover of $X$. Intersect all open sets $W \in \mathcal{W}$ with all $\bar{A}_{i+2} - A_i$, to construct a new open cover $\mathcal{W}'$ that refines $\mathcal{W}$. For each $x \in X$, find a coordinate neighborhood $U_x$ contained in some element of $\mathcal{W}'$, together with a surjective local coordinate $\varphi_x : U_x \to B^n_2$. Define $U'_x := \varphi_x^{-1}(B^n_1)$.

For each $i$, choose a cover of the compact set $A_i - \bar{A}_{i-1}$ by finitely many of the $U'_x, x \in Z_i$, where $Z_i \subset A_i - \bar{A}_{i-1}$ is a finite set. For convenience, set...
$Z_i := \emptyset$ for $i < 0$. The set $Z := \cup_i Z_i$ is countable; consider the cover $\mathcal{U}$ by all $U_x, x \in Z$. This cover is a refinement of $\mathcal{W}$, and it is locally finite: the open sets $A_{i+1} - A_i$ form an open cover of $X$, and each one can intersect only the finitely many

$$U_x, \quad \text{for } x \in \bigcup_{j=i-2}^{i+2} Z_j.$$  \hfill A1.4

Clearly our open sets are coordinate neighborhoods, as desired. \hfill \square

**Construction of partitions of unity**

It is now easy to construct partitions of unity subordinate to any cover of a second-countable $n$-dimensional manifold $X$.

**Theorem A1.7 (Partitions of unity)** Let $X$ be a second countable finite-dimensional topological manifold, and let $\mathcal{W}$ be an open cover of $X$. Then there exists a locally finite refinement $\mathcal{U}$ of $\mathcal{W}$, and a continuous partition of unity subordinate to $\mathcal{U}$. Moreover, if $X$ is a $C^\infty$ manifold, the partition of unity can be chosen $C^\infty$.

**Proof** Find a locally finite cover $\mathcal{U} := (U_x)_{x \in Z}$, as in the proof of Proposition A1.6, together with coordinate maps $\varphi_x$, which can be chosen $C^\infty$ if $X$ is $C^\infty$.

Let $h \geq 0$ be a $C^\infty$ function on $\mathbb{R}^n$ with support in $B^n_2$ and strictly positive on $B^n_1$. Define $h_x$, for $x \in Z$, by $h_x := h \circ \varphi_x$; since $h_x$ has compact support, it is the restriction of a continuous function on $X$ ($C^\infty$ if $X$ is $C^\infty$) with support in $U_x$, which we will still denote by $h_x$. Now the function $g := \sum_{x \in Z} h_x$ is a strictly positive continuous function on $X$; the sum exists and is continuous ($C^\infty$ if $X$ is $C^\infty$) because the cover $\mathcal{U}$ is locally finite. The functions $g_x, x \in Z$, defined by $g_x := h_x/g$ form the desired continuous partition of unity, $C^\infty$ if $X$ is $C^\infty$. \hfill \square