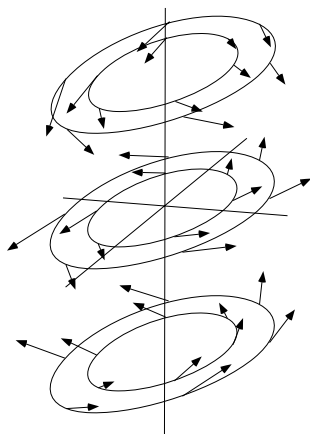
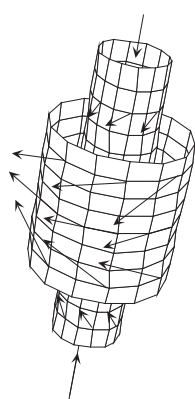
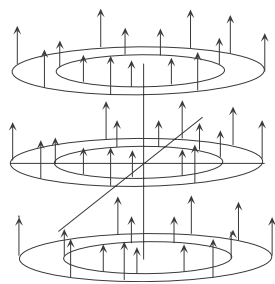


SOLUTIONS FOR CHAPTER 1



SOLUTION 1.1.7. TOP: The vector field in part a points up. MIDDLE: The vector field in part b is rotation in the (x, y) -plane. BOTTOM: The vector field in part c spirals out in the (x, y) -plane.

1.1.1 a. $\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

b. $2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 3-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

d. $\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \vec{e}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

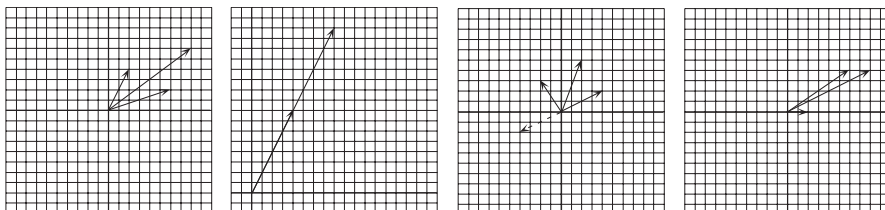


FIGURE FOR SOLUTION 1.1.1. From left: (a), (b), (c), and (d).

1.1.3 a. $\vec{v} \in \mathbb{R}^3$ b. $L \subset \mathbb{R}^2$ c. $C \subset \mathbb{R}^3$ d. $\mathbf{x} \in \mathbb{C}^2$
e. $B_0 \subset B_1 \subset B_2, \dots$

1.1.5 $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix} = \sum_{i=1}^n \vec{e}_i$ $\begin{bmatrix} 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{bmatrix} = \sum_{i=1}^n i \vec{e}_i$ $\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \\ \vdots \\ n-1 \\ n \end{bmatrix} = \sum_{i=3}^n i \vec{e}_i$

1.1.7 As shown in the figure in the margin, the vector field in part a points straight up everywhere. Its length depends only on how far you are from the z -axis, and it gets longer and longer the further you get from the z -axis; it vanishes on the z -axis. The vector field in part b is simply rotation in the (x, y) -plane, like (f) in Exercise 1.1.6. But the z -component is down when $z > 0$ and up when $z < 0$. The vector field in part c spirals out in the (x, y) -plane, like (h) in Exercise 1.1.6. Again, the z -component is down when $z > 0$ and up when $z < 0$.

1.1.9 The number 0 is in the set, since $\operatorname{Re}(w0) = 0$. If a and b are in the set, then $a+b$ is also in the set, since $\operatorname{Re}(w(a+b)) = \operatorname{Re}(wa) + \operatorname{Re}(wb) = 0$. If a is in the set and c is a real number, then ca is in the set, since then $\operatorname{Re}(wca) = c\operatorname{Re}(wa) = 0$. So the set is a subspace of $\mathbb{C}$. The subspace is a line in $\mathbb{C}$ with a polar angle θ such that $\theta + \varphi = k\pi/2$, where φ is the polar angle of w and k is an odd integer.

1.2.1 a. i. 2×3 ii. 2×2 iii. 3×2 iv. 3×4 v. 3×3

b. The matrices i and v can be multiplied on the right by the matrices iii, iv, v; the matrices ii and iii on the right by the matrices i and ii.

1.2.3 a. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ b. $\begin{bmatrix} 6 & 16 & 2 \end{bmatrix}$

1.2.5 a. True: $(AB)^\top = B^\top A^\top = B^\top A$

b. True: $(A^\top B)^\top = B^\top (A^\top)^\top = B^\top A = B^\top A^\top$

c. False: $(A^\top B)^\top = B^\top (A^\top)^\top = B^\top A \neq BA$

d. False: $(AB)^\top = B^\top A^\top \neq A^\top B^\top$

1.2.7 The matrices a and d have no transposes here. The matrices b and f are transposes of each other. The matrices c and e are transposes of each other.

1.2.9 a. $A^\top = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B^\top = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

b. $(AB)^\top = B^\top A^\top = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

c. $(AB)^\top = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}^\top = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

d. The matrix multiplication $A^\top B^\top$ is impossible.

1.2.11 The expressions b, c, d, f, g, and i make no sense.

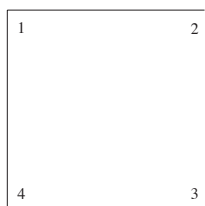
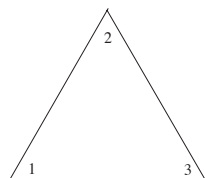
1.2.13 The trivial case is when $a = b = c = d = 0$; then obviously $ad - bc = 0$ and the matrix is not invertible. Let us suppose $d \neq 0$. (If we suppose that any other entry is nonzero, the proof would work the same way.) If $ad = bc$, then the first row is a multiple of the second: we can write $a = \frac{b}{d}c$ and $b = \frac{b}{d}d$, so the matrix is $A = \begin{bmatrix} \frac{b}{d}c & \frac{b}{d}d \\ c & d \end{bmatrix}$.

To show that A is not invertible, we need to show that there is no matrix $B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. But if the upper left corner of AB is 1, then we have $\frac{b}{d}(a'c + c'd) = 1$, so the lower left corner, which is $a'c + c'd$, cannot be 0.

1.2.15 $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & az+b+y \\ 0 & 1 & c+z \\ 0 & 0 & 1 \end{bmatrix}$

So $x = -a$, $z = -c$, and $y = ac - b$.

1.2.17 With the labeling shown in the margin, the adjacency matrices are



Labeling for Solution 1.2.17

$$\text{a.} \quad A_T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad A_S = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{b.} \quad A_T^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad A_T^3 = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} \quad A_T^4 = \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix}$$

$$A_T^5 = \begin{bmatrix} 10 & 11 & 11 \\ 11 & 10 & 11 \\ 11 & 11 & 10 \end{bmatrix}$$

$$A_S^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \quad A_S^3 = \begin{bmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{bmatrix} \quad A_S^4 = \begin{bmatrix} 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \\ 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \end{bmatrix}$$

$$A_S^5 = \begin{bmatrix} 0 & 16 & 0 & 16 \\ 16 & 0 & 16 & 0 \\ 0 & 16 & 0 & 16 \\ 16 & 0 & 16 & 0 \end{bmatrix}$$

The diagonal entries of A^n are the number of walks we can take of length n that take us back to our starting point.

$$A_n = \begin{bmatrix} a_n & b_n & b_n \\ b_n & a_n & b_n \\ b_n & b_n & a_n \end{bmatrix}$$

c. Let A_n be the matrix in the margin, so that $(A_n)_{i,j}$ is the number of walks of length n from vertex i to vertex j . (There are only two numbers by symmetry: there is nothing to distinguish one vertex from another and nothing to distinguish one pair of vertices from another.) To compute a_{n+1} and b_{n+1} we compute $A_n A_T = A_{n+1}$:

$$\begin{bmatrix} a_n & b_n & b_n \\ b_n & a_n & b_n \\ b_n & b_n & a_n \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2b_n & a_n + b_n & a_n + b_n \\ a_n + b_n & 2b_n & a_n + b_n \\ a_n + b_n & a_n + b_n & 2b_n \end{bmatrix},$$

so $a_{n+1} = 2b_n$ and $b_{n+1} = a_n + b_n$.

d. Color two opposite vertices of the square black and the other two white. Every move takes you from a vertex to a vertex of the opposite color.

e. Suppose such a coloring in black and white exists; then every walk goes from black to white to black to white $\dots$, in particular the (B, B) and the (W, W) entries of A^n are 0 for all odd n , and the (B, W) and (W, B) entries are 0 for all even n . Moreover, since the graph is connected, for any pair of vertices there is a walk of some length m joining them, and then

Part d: If you start at time 0 on black, you will be on black at all even times, and on white at all odd times, and there will be no walks of odd length from a vertex to itself.

the corresponding entry is nonzero for $m, m+2, m+4, \dots$ since you can go from the point of departure to the point of arrival in time m , and then bounce back and forth between this vertex and one of its neighbors.

Conversely, suppose the entries of A^n are zero or nonzero as described, and look at the top line of A^n , where n is chosen sufficiently large so that any entry that is ever nonzero is nonzero for A^{n-1} or A^n . The entries correspond to pairs of vertices (V_1, V_i) ; color in white the vertices V_i for which the $(1, i)$ entry of A^n is zero, and in black those for which the $(1, i)$ entry of A^{n+1} is zero. By hypothesis, we have colored all the vertices. It remains to show that adjacent vertices have different colors. Take a path of length m from V_1 to V_i . If V_j is adjacent to V_i , then there certainly exists a path of length $m+1$ from V_1 to V_j , namely the previous path, extended by one to go from V_i to V_j . Thus V_i and V_j have opposite colors.

Solution 1.2.19, part b: There are three ways to go from V_i to an adjacent vertex V_j and back again in three steps: go, return, stay; go, stay, return; and stay, go, return. Each vertex V_i has three adjacent vertices that can play the role of V_j . That brings us to nine. The couch potato itinerary “stay, stay, stay” brings us to 10.

1.2.19

$$\text{a. } B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{b. } B^2 = \begin{bmatrix} 4 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 4 & 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 4 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 4 & 2 & 2 & 0 & 2 \\ 2 & 0 & 2 & 2 & 4 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 4 & 2 \\ 0 & 2 & 2 & 2 & 2 & 2 & 2 & 4 \end{bmatrix} \quad B^3 = \begin{bmatrix} 10 & 10 & 6 & 10 & 6 & 10 & 6 & 6 \\ 10 & 10 & 10 & 6 & 6 & 6 & 10 & 6 \\ 6 & 10 & 10 & 10 & 6 & 6 & 6 & 10 \\ 10 & 6 & 10 & 10 & 10 & 6 & 6 & 6 \\ 6 & 6 & 6 & 10 & 10 & 10 & 6 & 10 \\ 10 & 6 & 6 & 6 & 10 & 10 & 10 & 6 \\ 6 & 10 & 6 & 6 & 6 & 10 & 10 & 10 \\ 6 & 6 & 10 & 6 & 10 & 6 & 10 & 10 \end{bmatrix}$$

The diagonal entries of B^3 correspond to the fact that there are exactly 10 loops of length 3 going from any given vertex back to itself.

Solution 1.2.21, part b: The first column of an adjacency matrix corresponds to vertex 1, the second to vertex 2, and so on, and the same for the rows. If the matrix is upper triangle, for example,

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ you can go from}$$

vertex 1 to vertex 2, but not from 2 to 1; from 2 to 3, but not from 3 to 2, and so on: once you have gone from a lower-numbered vertex to a higher-numbered vertex, there is no going back.

1.2.21 a. The proof is the same as with unoriented walks (Proposition 1.2.23): first we state that if B_n is the $n \times n$ matrix whose i, j th entry is the number of walks of length n from V_i to V_j , then $B_1 = A^1 = A$ for the same reasons as in the proof of Proposition 1.2.23. Here again if we assume the proposition true for n , we have:

$$(B_{n+1})_{i,j} = \sum_{k=1}^n (B_n)_{i,k} (B_1)_{k,j} = \sum_{k=1}^n (A^n)_{i,k} A_{k,j} = (A^{n+1})_{i,j}.$$

So $A^n = B_n$ for all n .

b. If the adjacency matrix is upper triangular, then you can only go from a lower number vertex to a higher number vertex; if it is lower triangular, you can only go from a higher number vertex to a lower number vertex. If it is diagonal, you can never go from any vertex to any other.

1.2.23 a. Let $A = \begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $AB = I$.

b. Whatever matrix one multiplies B by on the right, the top left corner of the resultant matrix will always be 0 when we need it to be 1. So the matrix B has no right inverse.

c. With A and B as in part a, write

$$I^\top = I = AB = (AB)^\top = B^\top A^\top.$$

So $A^\top$ is a right inverse for $B^\top$, so $B^\top$ has infinitely many right inverses.

1.3.1 a. Every linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ is given by a 2×4 matrix. For example, $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 7 \end{bmatrix}$ is a linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$.

b. Any row matrix 3 wide will do, for example, $[1, -1, 2]$; such a matrix takes a vector in $\mathbb{R}^3$ and gives a number.

1.3.3 A. $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ B. $\mathbb{R}^2 \rightarrow \mathbb{R}^5$ C. $\mathbb{R}^4 \rightarrow \mathbb{R}^2$ D. $\mathbb{R}^4 \rightarrow \mathbb{R}$.

$$\begin{bmatrix} .25 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .5 \end{bmatrix}, \text{ putting } \vec{v}$$

1.3.5 Multiply the original matrix by the vector $\vec{v} =$

on the right.

1.3.7 It is enough to know what T gives when evaluated on the three standard basis vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The matrix of T is $\begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

1.3.9 No, T is not linear. If it were linear, the matrix would be $[T] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, but $[T] \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 5 \end{bmatrix}$, which contradicts the definition of the transformation.

Solution 1.3.1: The first question on one exam at Cornell University was “What is a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$?” Several students gave answers like “A function taking some vector $\vec{v} \in \mathbb{R}^n$ and producing some vector $\vec{w} \in \mathbb{R}^m$.”

This is wrong! The student who gave this answer may have been thinking of matrix multiplication. But a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ need not be given by matrix multiplication: consider the mapping $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$. In any case, defining a linear transformation by saying that it is given by matrix multiplication begs the question, because it does not explain why matrix multiplication is linear.

The correct answer was given by another student in the class:

“A linear transformation

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

is a mapping $\mathbb{R}^n \rightarrow \mathbb{R}^m$ such that for all $\vec{a}, \vec{b} \in \mathbb{R}^n$,

$$T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b}),$$

and for all $\vec{a} \in \mathbb{R}^n$ and all scalar r ,

$$T(r\vec{a}) = rT(\vec{a}).”$$

Linearity and the approximation of nonlinear mappings by linear mappings are key motifs of this book. You must know the definition, which gives you a foolproof way to check whether or not a given mapping is linear.

1.3.11 The rotation matrix is $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$. This transformation takes $\vec{e}_1$ to $\begin{bmatrix} \cos \theta \\ -\sin \theta \end{bmatrix}$, which is thus the first column of the matrix, by Theorem 1.3.4; it takes $\vec{e}_2$ to $\begin{bmatrix} \cos(90^\circ - \theta) = \sin \theta \\ \sin(90^\circ - \theta) = \cos \theta \end{bmatrix}$, which is the second column.

One could also write this mapping as the rotation matrix of Example 1.3.9, applied to $-\theta$:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}.$$

1.3.13 The expressions a, e, f, and j are not well-defined compositions. For the others:

- b. $C \circ B : \mathbb{R}^m \rightarrow \mathbb{R}^n$ (domain $\mathbb{R}^m$, codomain $\mathbb{R}^n$) c. $A \circ C : \mathbb{R}^k \rightarrow \mathbb{R}^m$
d. $B \circ A \circ C : \mathbb{R}^k \rightarrow \mathbb{R}^k$ g. $B \circ A : \mathbb{R}^n \rightarrow \mathbb{R}^k$
h. $A \circ C \circ B : \mathbb{R}^m \rightarrow \mathbb{R}^m$ i. $C \circ B \circ A : \mathbb{R}^n \rightarrow \mathbb{R}^n$

1.3.15 We need to show that $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$ and $A(c\vec{v}) = cA\vec{v}$. By Definition 1.2.4,

Solution 1.3.15: With the dot product, introduced in the next section, the solution is simpler: Every entry in the vector $A\vec{v}$ is the dot product of $\vec{v}$ with one of the rows of A . The dot product is linear with respect to both of its arguments, so the mapping $A\vec{v}$ is also linear with respect to $\vec{v}$.

$$\begin{aligned} (A\vec{v})_i &= \sum_{k=1}^n a_{i,k} v_k, & (A\vec{w})_i &= \sum_{k=1}^n a_{i,k} w_k, \text{ and} \\ (A(\vec{v} + \vec{w}))_i &= \sum_{k=1}^n a_{i,k} (v + w)_k = \sum_{k=1}^n a_{i,k} (v_k + w_k) \\ &= \sum_{k=1}^n a_{i,k} v_k + \sum_{k=1}^n a_{i,k} w_k = (A\vec{v})_i + (A\vec{w})_i. \end{aligned}$$

$$\text{Similarly, } (A(c\vec{v}))_i = \sum_{k=1}^n a_{i,k} (cv)_k = \sum_{k=1}^n a_{i,k} c v_k = c \sum_{k=1}^n a_{i,k} v_k = c(A\vec{v})_i.$$

1.3.17

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}^2 = \begin{bmatrix} \cos(2\theta)^2 + \sin(2\theta)^2 & \cos(2\theta)\sin(2\theta) - \sin(2\theta)\cos(2\theta) \\ \sin(2\theta)\cos(2\theta) - \cos(2\theta)\sin(2\theta) & \sin(2\theta)^2 + \cos(2\theta)^2 \end{bmatrix}$$

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

1.3.19 By commutativity of matrix addition, $\frac{AB+BA}{2} = \frac{BA+AB}{2}$, so the Jordan product is commutative. By non-commutativity of matrix multiplication:

$$\begin{aligned} \frac{\frac{AB+BA}{2}C + C\frac{AB+BA}{2}}{2} &= \frac{ABC + BAC + CAB + CBA}{4} \\ &\neq \frac{ABC + ACB + BCA + CBA}{4} = \frac{A\frac{BC+CB}{2} + \frac{BC+CB}{2}A}{2} \end{aligned}$$

so the Jordan product is not associative.

1.4.1 a. Numbers: $\vec{v} \cdot \vec{w}$, $|\vec{v}|$, $|A|$, and $\det A$. (If A consists of a single row, then $A\vec{v}$ is also a number.)

Vectors: $\vec{v} \times \vec{w}$ and $A\vec{v}$ (unless A consists of a single row).

So far we have defined only determinants of 2×2 and 3×3 matrices; in Section 4.8 we will define the determinant in general.

b. In the expression $\vec{v} \times \vec{w}$, the vectors must each have three entries. In the expression $\det A$, the matrix A must be square.

1.4.3 To normalize a vector, divide it by its length. This gives:

$$\text{a. } \frac{1}{\sqrt{17}} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \quad \text{b. } \frac{1}{\sqrt{58}} \begin{bmatrix} -3 \\ 7 \end{bmatrix} \quad \text{c. } \frac{1}{\sqrt{31}} \begin{bmatrix} \sqrt{2} \\ -2 \\ -5 \end{bmatrix}$$

$$\text{1.4.5 a. } \cos(\theta) = \frac{1}{1 \times \sqrt{3}} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \frac{1}{\sqrt{3}}, \text{ so}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{3}}\right) \approx .95532.$$

$$\text{b. } \cos(\theta) = 0, \text{ so } \theta = \pi/2.$$

$$\text{1.4.7 a. } \det = 1; \text{ the inverse is } \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\text{b. } \det = 0; \text{ no inverse}$$

$$\text{c. } \det = ad; \text{ if } a, d \neq 0, \text{ the inverse is } \frac{1}{ad} \begin{bmatrix} d & -b \\ 0 & a \end{bmatrix}.$$

$$\text{d. } \det = 0; \text{ no inverse}$$

$$\text{1.4.9 a. } \begin{bmatrix} -6yz \\ 3xz \\ 5xy \end{bmatrix} \quad \text{b. } \begin{bmatrix} 6 \\ 7 \\ -4 \end{bmatrix} \quad \text{c. } \begin{bmatrix} -2 \\ -22 \\ 3 \end{bmatrix}$$

1.4.11 a. True by Theorem 1.4.5, because $\vec{w} = -2\vec{v}$.

Solution 1.4.11, part c: Our answer depended on the vectors chosen. In general,

$$\det[\vec{a}, \vec{b}, \vec{c}] = -\det[\vec{a}, \vec{c}, \vec{b}];$$

the result here is true only because both sides are 0, since $\vec{v}, \vec{w}$ are linearly dependent.

b. False; $\vec{u} \cdot (\vec{v} \times \vec{w})$ is a number; $|\vec{u}|(\vec{v} \times \vec{w})$ is a number times a vector, i.e., a vector.

c. True: since $\vec{w} = -2\vec{v}$, we have $\det[\vec{u}, \vec{v}, \vec{w}] = 0$ and $\det[\vec{u}, \vec{w}, \vec{v}] = 0$.

d. False, since $\vec{u}$ is not necessarily (in fact almost surely not) a multiple of $\vec{w}$; the correct statement is $|\vec{u} \cdot \vec{w}| \leq |\vec{u}||\vec{w}|$.

e. True. f. True.

$$\text{1.4.13 } \begin{bmatrix} xa \\ xb \\ xc \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} xbc - xbc \\ -(xac - xac) \\ xab - xab \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solutions 1.4.13 and 1.4.15: Of course, the vectors must be in $\mathbb{R}^3$ for the cross product to be defined.

$$\text{1.4.15 } \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} bf - ce \\ cd - af \\ ae - bd \end{bmatrix} = - \begin{bmatrix} d \\ e \\ f \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = - \begin{bmatrix} ec - bf \\ af - cd \\ bd - ae \end{bmatrix}$$

1.4.17 a. It is the line of equation $\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2x - y = 0$.

b. It is the line of equation $\begin{bmatrix} x-2 \\ y-3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \end{bmatrix} = 2x - 4 - 4y + 12 = 0$, which you can rewrite as $2x - 4y + 8 = 0$.

1.4.19 a. The length of $\vec{v}_n$ is $|\vec{v}_n| = \sqrt{1 + \cdots + 1} = \sqrt{n}$.

b. The angle is $\arccos \frac{1}{\sqrt{n}}$, which tends to $\pi/2$ as $n \rightarrow \infty$.

Solution 1.4.19, part b: We find it surprising that the diagonal vector $\vec{v}_n$ is almost orthogonal to all the standard basis vectors when n is large.

1.4.21 a. $|A| = \sqrt{1+1+4} = \sqrt{6}$; $|B| = \sqrt{5}$; $|\vec{c}| = \sqrt{10}$

b. $|AB| = \left| \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \right| = \sqrt{12} \leq \sqrt{30} = |A||B|$

$|A\vec{c}| = \sqrt{50} \leq |A||\vec{c}| = \sqrt{60}$; $|B\vec{c}| = \sqrt{13} \leq |B||\vec{c}| = \sqrt{50}$

1.4.23 a. The length is $|\vec{w}_n| = \sqrt{1+4+\cdots+n^2} = \sqrt{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}}$.

b. The angle $\alpha_{n,k}$ is $\arccos \frac{k}{\sqrt{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}}}$.

Solution 1.4.23, part a: This uses the fact that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

c. In all three cases, the limit is $\pi/2$. Clearly $\lim_{n \rightarrow \infty} \alpha_{n,k} = \pi/2$, since the cosine tends to 0.

The limit of $\alpha_{n,n}$ is $\lim_{n \rightarrow \infty} \arccos \frac{n}{\sqrt{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}}} = \arccos 0 = \pi/2$.

The limit of $\alpha_{n,[n/2]}$ is also $\pi/2$, since it is the arccos of

$$\frac{[n/2]}{\sqrt{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}}}, \quad \text{which tends to 0 as } n \rightarrow \infty.$$

1.4.25

$$\begin{aligned} \left(\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2} \right)^2 - (x_1 y_1 + x_2 y_2)^2 &= (x_2 y_1)^2 + (x_1 y_2)^2 - 2x_1 x_2 y_1 y_2 \\ &= (x_1 y_2 - x_2 y_1)^2 \geq 0, \end{aligned}$$

so $(x_1 y_1 + x_2 y_2)^2 \leq \left(\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2} \right)^2$, so

$$|x_1 y_1 + x_2 y_2| \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}.$$

1.4.27 a. To show that the transformation is linear, we need to show that

$$T_{\vec{a}}(\vec{v} + \vec{w}) = T_{\vec{a}}(\vec{v}) + T_{\vec{a}}(\vec{w}) \quad \text{and} \quad \alpha T_{\vec{a}}(\vec{v}) = T_{\vec{a}}(\alpha \vec{v}).$$

For the first,

$$T_{\vec{a}}(\vec{v} + \vec{w}) = \vec{v} + \vec{w} - 2(\vec{a} \cdot (\vec{v} + \vec{w}))\vec{a} = \vec{v} + \vec{w} - 2(\vec{a} \cdot \vec{v} + \vec{a} \cdot \vec{w})\vec{a} = T_{\vec{a}}(\vec{v}) + T_{\vec{a}}(\vec{w}).$$

For the second,

$$\alpha T_{\vec{a}}(\vec{v}) = \alpha \vec{v} - 2\alpha(\vec{a} \cdot \vec{v})\vec{a} = \alpha \vec{v} - 2(\vec{a} \cdot \alpha \vec{v})\vec{a} = T_{\vec{a}}(\alpha \vec{v}).$$

b. We have $T_{\vec{a}}(\vec{a}) = -\vec{a}$, since $\vec{a} \cdot \vec{a} = a^2 + b^2 + c^2 = 1$:

$$T_{\vec{a}}(\vec{a}) = \vec{a} - 2(\vec{a} \cdot \vec{a})\vec{a} = \vec{a} - 2\vec{a} = -\vec{a}.$$

If $\vec{v}$ is orthogonal to $\vec{a}$, then $T_{\vec{a}}(\vec{v}) = \vec{v}$, since in that case $\vec{a} \cdot \vec{v} = 0$. Thus $T_{\vec{a}}$ is reflection in the plane that goes through the origin and is perpendicular to $\vec{a}$.

c. The matrix of $T_{\vec{a}}$ is

$$M = [T_{\vec{a}}(\vec{e}_1), T_{\vec{a}}(\vec{e}_2), T_{\vec{a}}(\vec{e}_3)] = \begin{bmatrix} 1 - 2a^2 & -2ab & -2ac \\ -2ab & 1 - 2b^2 & -2bc \\ -2ac & -2bc & 1 - 2c^2 \end{bmatrix}$$

Squaring the matrix gives the 3×3 identity matrix: if you reflect a vector, then reflect it again, you are back to where you started.

The transformation $T_{\vec{a}}$ is the 3-dimensional version of the transformation shown in Figure 1.3.4.

Solution 1.5.1: Note that part c is the same as part a.

1.5.1 a. The set $\{x \in \mathbb{R} \mid 0 < x \leq 1\}$ is neither open nor closed: the point 1 is in the set, but $1 + \epsilon$ is not for every ϵ , showing it isn't open, and 0 is not but $0 + \epsilon$ is for every $\epsilon > 0$, showing that the complement is also not open, so the set is not closed.

b. open c. neither d. closed e. closed f. neither

g. Both. That the empty set $\emptyset$ is closed is obvious. Showing that it is open is an “eleven-legged alligator” argument (see Section 0.2). Is it true that for every point of the empty set, there exists $\epsilon > 0$ such that $\dots$? Yes, because there are no points in the empty set.

1.5.3 a. Suppose $A_i, i \in I$ is some collection (probably infinite) of open sets. If $\mathbf{x} \in \bigcup_{i \in I} A_i$, then $\mathbf{x} \in A_j$ for some j , and since A_j is open, there exists $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset A_j$. But then $B_\epsilon(\mathbf{x}) \subset \bigcup_{i \in I} A_i$.

Solution 1.5.3, part b: There is a smallest ϵ_i , because there are finitely many of them, and it is positive. If there were infinitely many, then there would be a greatest lower bound, but it could be 0.

Part c: In fact, every closed set is a countable intersection of open sets.

b. If $A_1, \dots, A_j$ are open and $\mathbf{x} \in \bigcap_{i=1}^k A_i$, then there exist $\epsilon_1, \dots, \epsilon_k > 0$ such that $B_{\epsilon_i}(\mathbf{x}) \subset A_i$, for $i = 1, \dots, k$. Set ϵ to be the smallest of $\epsilon_1, \dots, \epsilon_k$. Then $B_\epsilon(\mathbf{x}) \subset B_{\epsilon_i}(\mathbf{x}) \subset A_i$.

c. The infinite intersection of open sets $(-1/n, 1/n)$, for $n = 1, 2, \dots$, is not open; as $n \rightarrow \infty$, $-1/n \rightarrow 0$ and $1/n \rightarrow 0$; the set $\{0\}$ is not open.

1.5.5 a. This set is open. Indeed, if you choose $\begin{pmatrix} x \\ y \end{pmatrix}$ in your set, then $1 < \sqrt{x^2 + y^2} < \sqrt{2}$. Set

$$r = \min \left\{ \sqrt{x^2 + y^2} - 1, \sqrt{2} - \sqrt{x^2 + y^2} \right\} > 0.$$

Equation 1 uses the familiar form of the triangle inequality: if $\mathbf{a} = \mathbf{b} + \mathbf{c}$, then

$$|\mathbf{a}| \leq |\mathbf{b}| + |\mathbf{c}|.$$

Equation 2 uses the variant

$$|\mathbf{a}| \geq \left| |\mathbf{b}| - |\mathbf{c}| \right|.$$

Then the ball of radius r around $\begin{pmatrix} x \\ y \end{pmatrix}$ is contained in the set, since if $\begin{pmatrix} u \\ v \end{pmatrix}$ is in that ball, then, by the triangle inequality,

$$\left| \begin{bmatrix} u \\ v \end{bmatrix} \right| \leq \left| \begin{bmatrix} u-x \\ v-y \end{bmatrix} \right| + \left| \begin{bmatrix} x \\ y \end{bmatrix} \right| < r + \left| \begin{bmatrix} x \\ y \end{bmatrix} \right| \leq \sqrt{2} \quad (1)$$

$$\left| \begin{bmatrix} u \\ v \end{bmatrix} \right| \geq \left| \begin{bmatrix} x \\ y \end{bmatrix} \right| - \left| \begin{bmatrix} u-x \\ v-y \end{bmatrix} \right| > \left| \begin{bmatrix} x \\ y \end{bmatrix} \right| - r \geq 1. \quad (2)$$

b. The locus $xy \neq 0$ is also open. It is the complement of the two axes, so that if $\begin{pmatrix} x \\ y \end{pmatrix}$ is in the set, then $r = \min\{|x|, |y|\} > 0$, and the ball B of radius r around $\begin{pmatrix} x \\ y \end{pmatrix}$ is contained in the set. Indeed, if $\begin{pmatrix} u \\ v \end{pmatrix}$ is B , then $|u| = |x + u - x| > |x| - |u - x| > |x| - r \geq 0$, so u is not 0, and neither is v , by the same argument.

c. This time our set is the x -axis, and it is closed. We will use the criterion that a set is closed if the limit of a convergent sequence of elements of the set is in the set (Proposition 1.5.17). If $n \mapsto \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ is a sequence in the set, and converges to $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, then all $y_n = 0$, so $y_0 = \lim_{n \rightarrow \infty} y_n = 0$, and the limit is also in the set.

d. The rational numbers are neither open nor closed. Any rational number x is the limit of the numbers $x + \sqrt{2}/n$, which are all irrational, so the irrationals aren't closed. Any irrational number is the limit of the finite decimals used to write it, which are all rational, so the rationals aren't closed either.

Many students have found Exercise 1.5.5 difficult, even though they also thought it was obvious, but didn't know how to say it. If this applies to you, you should check carefully where we used the triangle inequality, and how. Almost everything concerning inequalities requires the triangle inequality.

1.5.7 a. The natural domain is $\mathbb{R}^2$ minus the union of the two axes; it is open.

b. The natural domain is that part of $\mathbb{R}^2$ where $x^2 > y$ (i.e., the area “outside” the parabola of equation $y = x^2$). It is open since its “fence” x^2 belongs to its neighbor.

c. The natural domain of $\ln \ln x$ is $\{x | x > 1\}$, since we must have $\ln x > 0$. This domain is open.

d. The natural domain of $\arcsin$ is $[-1, 1]$. Thus the natural domain of $\arcsin \frac{3}{x^2+y^2}$ is $\mathbb{R}^2$ minus the open disc $x^2 + y^2 < 3$. Since this domain is the complement of an open disc it is closed (and not open, since it isn't $\mathbb{R}^2$ or the empty set).

e. The natural domain is all of $\mathbb{R}^2$, which is open.

f. The natural domain is $\mathbb{R}^3$ minus the union of the three coordinate planes of equation $x = 0$, $y = 0$, $z = 0$; it is open.

1.5.9 For any $n > 0$ we have $\left| \sum_{i=1}^n \mathbf{x}_i \right| \leq \sum_{i=1}^n |\mathbf{x}_i|$ by the triangle inequality (Theorem 1.4.9). Because $\sum_{i=1}^{\infty} \mathbf{x}_i$ converges, $\sum_{i=1}^n \mathbf{x}_i$ converges as $n \rightarrow \infty$. So :

$$\left| \sum_{i=1}^{\infty} \mathbf{x}_i \right| \leq \sum_{i=1}^{\infty} |\mathbf{x}_i|.$$

1.5.11 a. First let us see that

$$\left((\forall \epsilon > 0)(\exists N)(n > N) \implies |\mathbf{a}_n - \mathbf{a}| < \varphi(\epsilon) \right) \implies (\mathbf{a}_n \text{ converges to } \mathbf{a}).$$