

Contents

0	Three important inequalities	1
1	Metric and topological spaces	6
1.1	Metrics and metric spaces	6
1.2	Open and closed sets	16
1.3	Topological spaces	25
1.4	Continuous functions	33
1.5	Open sets and continuity	36
1.6	Some important topological concepts	45
1.7	Convergence of sequences in metric spaces	50
1.8	Completeness	67
1.9	Density, separability, and approximation	73
1.10	Metric space completions	81
1.11	Compactness	88
1.12	The Banach fixed point theorem	102
1.13	Baire's category theorem	107
2	Normed spaces	116
2.1	Linear operators on function spaces	116
2.2	Hamel bases	120
2.3	Norms and normed spaces	125
2.4	Topological concepts in normed spaces	137
2.5	Topological vector spaces	144
2.6	Kolmogorov's theorem	157
2.7	Banach spaces	165
2.8	Infinite series in normed spaces	175
2.9	Schauder bases	179
2.10	Linear functionals and hyperplanes	186
2.11	Constructing new normed spaces	199
3	Operators on normed spaces	206
3.1	Continuous linear maps	206
3.2	Integral operators	219
3.3	Linear homeomorphisms	229

3.4	Three important theorems	242
3.5	The normed space $\mathcal{B}(X, Y)$	256
3.6	Complementary subspaces and projections	268
3.7	Riesz's lemma	274
3.8	The spectrum of a bounded linear operator	279
3.9	Continuous linear functionals and dual spaces	285
4	Inner product spaces	291
4.1	Definitions and examples	291
4.2	Orthogonality	303
4.3	Unitary isomorphisms	311
4.4	Inner product spaces: three problems	317
4.5	Three characterizations for Hilbert spaces	328
4.6	Hilbert bases	342
5	The Banach space $C(X)$	365
5.1	The Arzela-Ascoli theorem	365
5.2	Korovkin's theorem and the Weierstrass approximation theorem	374
5.3	Sub-algebras of $C(X, \mathcal{K})$	387
5.4	The Stone-Weierstrass theorem	397
6	Additional Topics	415
6.1	The Baire-Osgood theorem	415
6.2	Gram determinants and Muntz's theorem	424
6.3	Differential equations	438
	Appendices	464
A	Set theory and functions	464
A.1	Sets	464
A.2	Relations	466
A.3	Zorn's lemma and the axiom of choice	469
A.4	Functions	470
A.5	Cardinality	471
A.6	The axiom of completeness on \mathbb{R}	477
B	Mostly linear algebra: a brief review	480
B.1	Polynomials and sequences	480
B.2	Vector spaces	481
B.3	Linear independence and span	483
B.4	Bases and dimensions	485
B.5	Linear transformations	486
B.6	Partial derivatives and the mean value theorem	488
B.7	Riemann integrals	490

C	Some technical results	491
C.1	Lemma used to prove Theorem 4.1.14	491
C.2	Lemma used to prove Proposition 6.2.5	492
D	Solutions to odd exercises	494
D.1	Solutions for Chapter 1	494
D.2	Solutions for Chapter 2	515
D.3	Solutions for Chapter 3	547
D.4	Solutions for Chapter 4	575
D.5	Solutions for Chapter 5	595
D.6	Solutions for Chapter 6	611
	Bibliography	618
	Notation	622
	Index	624