A fixed point theorem for integral equations

The fundamental theorem of calculus provides a close relationship between differential equations and integral equations. The main result in this subsection is Theorem 6.3.12, which relates being Lipschitz to existence and uniqueness of solutions to certain integral equations. From it, a host of important results will follow.

The proof of Theorem 6.3.12 shows that it concerns the existence and uniqueness of a fixed point of a map on a set X; thus it uses the Banach fixed point theorem (Corollary 1.12.4). As completeness is one of the two main ingredients in the Banach fixed point theorem (contractions being the other), let us single out the complete metric space that is appropriate for our purpose.

Let I be a closed bounded interval in $\mathbb{R}$. By Corollary 1.11.18, $C(I, \mathbb{R}^n)$ is a complete metric space under the metric

$$\rho(\mathbf{g}, \mathbf{h}) := \max_{t \in I} \|\mathbf{g}(t) - \mathbf{h}(t)\|, \tag{6.3.19}$$

where $\| \ \|$ is the Euclidean norm on $\mathbb{R}^n$. By Corollary 3.3.5, the same is true if $\mathbb{R}^n$ is given any norm.

Lemma 6.3.11. *Let S be a closed subset of $\mathbb{R}^n$ and consider the metric subspace $X \subset C(I, \mathbb{R}^n)$ consisting of all $\mathbf{f} \in C(I, \mathbb{R}^n)$ with $\mathbf{f}(I) \subset S$. Then X is complete.*

Proof. Suppose $(\mathbf{f}_n) \subset X$ is Cauchy. Then since $C(I, \mathbb{R}^n)$ is complete, there is some $\mathbf{g} \in C(I, \mathbb{R}^n)$ such that

$$\lim_{n \to \infty} \max_{t \in I} \|\mathbf{f}_n(t) - \mathbf{g}(t)\| = 0. \tag{6.3.20}$$

Since $\mathbf{f}_n(I) \subset S$ for all n and S is closed, (6.3.20) implies that

$$\mathbf{g}(t) = \lim_{n \to \infty} \mathbf{f}_n(t) \quad \text{and thus} \quad \mathbf{g}(t) \in \overline{S} = S \quad \text{for all } t \in I. \tag{6.3.21}$$

So, $\mathbf{g} \in X$. Hence, X is a closed subset of $C(I, \mathbb{R}^n)$. Since $C(I, \mathbb{R}^n)$ is complete, X is also complete, by Proposition 1.8.8. □

Since X is complete, any contraction $T : X \to X$ has a unique fixed point, by the Banach fixed point theorem. To prove existence and uniqueness of solutions to systems of first-order differential equations, we will need an appropriate contraction on X. This is where Lipschitz functions come into play.

Theorem 6.3.12 (Solutions to integral equations). *Let $U \subset \mathbb{R}^n$ be open and $I \subset \mathbb{R}$. Let $\mathbf{f} : I \times U \to \mathbb{R}^n$ be continuous and Lipschitz with respect to the variable in U. If $(t_0, \mathbf{z_0})$ is an interior point of $I \times U$, there exists $\delta_0 > 0$ such that $[t_0 - \delta_0, t_0 + \delta_0] \subset I$ and for any δ satisfying $0 < \delta \leq \delta_0$, the equation*

$$\mathbf{x}(t) = \mathbf{z}_0 + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{x}(s))\, ds \tag{6.3.22}$$

has a unique continuous solution $\mathbf{u} : [t_0 - \delta, t_0 + \delta] \to U$.

Remarks.

1. Saying that $\mathbf{u}$ is a solution to (6.3.22) on $[t_0 - \delta, t_0 + \delta]$ means that

$$\mathbf{u}(t) = \mathbf{z}_0 + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{u}(s))\, ds \quad \text{for all } t \in [t_0 - \delta, t_0 + \delta]. \tag{6.3.23}$$

2. The point of having both δ and δ_0 is to ensure that the unique solution guaranteed on $[t_0 - \delta_0, t_0 + \delta_0] \subset I$ restricts to the unique solution on the (equal or smaller) subinterval $[t_0 - \delta, t_0 + \delta]$. $\triangle$

PROOF. Since $(t_0, \mathbf{z_0})$ is an interior point of $I \times U$, we have a closed ball $\overline{B(\mathbf{z}_0, r)}$ contained in U and a closed interval $[a, b] \subset I$ containing t_0 in its interior.

Since $\mathbf{f}$ is continuous on the compact set $[a, b] \times \overline{B(\mathbf{z}_0, r)}$ and is Lipschitz with respect to the variable in U, there are positive constants C_1, C_2 such that

$$\|\mathbf{f}(t, \mathbf{x})\| \le C_1, \quad t \in [a, b],\ \mathbf{x} \in \overline{B(\mathbf{z}_0, r)} \tag{6.3.24}$$

$$\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \le C_2 \|\mathbf{x}_1 - \mathbf{x}_2\|,\ t \in [a, b], \mathbf{x}_1, \mathbf{x}_2 \in U. \tag{6.3.25}$$

Put

$$\delta_0 := \min\left\{ \frac{1}{C_1},\ \frac{1}{2C_2},\ \frac{r}{C_1},\ |t_0 - a|,\ |t_0 - b| \right\}. \tag{6.3.26}$$

Let $0 < \delta \le \delta_0$. Set $J := [t_0 - \delta, t_0 + \delta]$. Then $J \subset [a, b]$.

Let X be the metric subspace of $C(J, \mathbb{R}^n)$ consisting of all continuous functions $\mathbf{g} : J \to \overline{B(\mathbf{z}_0, r)}$. For $\mathbf{g} \in X$, let

$$(T\mathbf{g})(t) := \mathbf{z}_0 + \int_{t_0}^{t} \mathbf{f}\big(s, \mathbf{g}(s)\big)\, ds \text{ for all } t \in J. \tag{6.3.27}$$

Note that any $\mathbf{u} \in X$ with $T\mathbf{u} = \mathbf{u}$ will be a solution to (6.3.22). Since X is complete (Lemma 6.3.11), the Banach fixed point theorem guarantees a unique fixed point for T if we can show that T is a contraction that takes X into X. First we will show that $T\mathbf{g}$ is in X: i.e., that it is continuous and goes from J to $\overline{B(\mathbf{z}_0, r)}$. Let $\mathbf{g} \in X$. Then $T\mathbf{g}$ is (uniformly) continuous on J, since for any $t_1, t_2 \in J$,

$$\begin{aligned} \|T\mathbf{g}(t_1) - T\mathbf{g}(t_2)\| &= \left\| \int_{t_1}^{t_2} \mathbf{f}(s, \mathbf{g}(s))\, ds \right\| \\ &\underbrace{\le}_{(2.7.10)} |t_1 - t_2| \sup_{s \in [t_1, t_2]} \|\mathbf{f}(s, \mathbf{g}(s))\| \underbrace{\le}_{(6.3.24)} C_1 |t_1 - t_2|. \end{aligned} \tag{6.3.28}$$

Also, $T\mathbf{g}(J) \subset \overline{B(\mathbf{z}_0, r)}$: if $t \in J$, then $|t - t_0| < \delta$ and thus

$$\|T\mathbf{g}(t) - \mathbf{z}_0\| = \left\| \int_{t_0}^{t} \mathbf{f}(s, \mathbf{g}(s))\, ds \right\| \underbrace{\le}_{(2.7.10)} C_1 |t - t_0| \le C_1 \delta \le C_1 \delta_0 \underbrace{\le}_{(6.3.26)} r.$$