Chapter 6

Additional topics

In this chapter we explore several interesting applications of results developed so far: the Baire-Osgood theorem, which uses Baire's category theorem; Muntz's theorem, which uses the formula for computing distances from a point to a finite-dimensional vector subspace of an inner product space; and the existence and uniqueness of solutions to differential equations, which use the Banach fixed point theorem. Completeness plays a crucial role in most of these applications.

6.1 The Baire-Osgood theorem

The Baire-Osgood theorem says that if a sequence of continuous functions from a Banach space X to $\mathbb{R}$ converges pointwise to a function f, then f must be continuous at "most" points in X. It is proved by showing that the set of points in X at which f is discontinuous is thin in X.

Notation 6.1.1 (Set of discontinuities). Let $f : (X, d) \to (Y, \rho)$ be a function. We denote by $D(f)$ the set of all points in X at which f is discontinuous.

This set $D(f)$ can be characterized nicely in terms of closed sets in X. This requires reformulating the condition of continuity. Recall from Definition 1.1.19 that $\operatorname{diam} A := \sup\{\, d(x, y) \mid x, y \in A \,\}$.

Definition 6.1.2 (Oscillation). Let $f : (X, d) \to (Y, \rho)$ be a function and let $A \neq \emptyset$ be a bounded subset in X. The *oscillation of f on A*, denoted by $\Omega(f; A)$, is

$$\Omega(f; A) := \operatorname{diam} f(A). \tag{6.1.1}$$

For $x \in X$, the *oscillation of f at x* is

$$\omega_f(x) := \inf_{\delta > 0} \Omega(f; B(x, \delta)). \tag{6.1.2}$$

It is obvious that if a nonempty set A is a subset of B, then we have $\Omega(f;A) \le \Omega(f;B)$; if $x \in X$, then for all $\eta > 0$,

$$0 \le \omega_f(x) \le \Omega\big(f; B(x,\eta)\big). \tag{6.1.3}$$

Proposition 6.1.3. *Let $f : (X,d) \to (Y,\rho)$ be a function. Then*

1. *For each $r > 0$, the set $\{\, a \in X \mid \omega_f(a) < r \,\}$ is open in X.*
2. *For each $a \in X$, f is continuous at a if and only if $\omega_f(a) = 0$.*

Proof. 1. Let $r > 0$ and set $U := \{\, a \in X \mid \omega_f(a) < r \,\}$. Assume that $x_0 \in U$. We will show that x_0 is an interior point of U. Since $\omega_f(x_0) < r$, there must be some $\delta > 0$ such that

$$\Omega\big(f; B(x_0,\delta)\big) < r. \tag{6.1.4}$$

Let $t \in B(x_0,\delta)$; then there is some $\eta > 0$ such that $t \in B(t,\eta) \subset B(x_0,\delta)$. Hence, $\omega_f(t) \le \Omega(f;B(t,\eta)) \le \Omega(f;B(x_0,\delta)) < r$. This shows that $t \in U$. Since $t \in B(x_0,\delta)$ was arbitrary, we conclude that $B(x_0,\delta) \subset U$.

2. Assume that f is continuous at $a \in X$. Let $\epsilon > 0$ be arbitrary. Then there is some $\delta > 0$ such that whenever $t \in B(a,\delta)$, we have $\rho(f(t), f(a)) < \frac{\epsilon}{2}$. Thus, for $x, y \in B(a,\delta)$,

$$\rho\big(f(x), f(y)\big) \le \rho\big(f(x), f(a)\big) + \rho\big(f(a), f(y)\big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{6.1.5}$$

Consequently, $\Omega(f;B(a,\delta)) \le \epsilon$. Thus, from (6.1.3), $0 \le \omega_f(a) \le \epsilon$. Because $\epsilon > 0$ was arbitrary, we must have

$$\omega_f(a) = 0. \tag{6.1.6}$$

Conversely, assume f is not continuous at a. Then there is some $\epsilon_0 \ge 0$ such that for any $\delta > 0$, there is some $t \in B(a,\delta)$ such that $\rho(f(a), f(t)) \ge \epsilon_0$. In particular, $\Omega(f;B(a,\delta)) \ge \epsilon_0$. Since $\delta > 0$ was arbitrary, it follows that

$$\omega_f(a) \ge \epsilon_0 > 0. \qquad \square \tag{6.1.7}$$

Corollary 6.1.4. *Let $f : X \to Y$ be a function between metric spaces. Then the set $D(f)$ of points in X at which f is discontinuous is*

$$D(f) = \bigcup_{n=1}^{\infty} \left\{ a \in X \;\middle|\; \omega_f(a) \ge \frac{1}{n} \right\}. \tag{6.1.8}$$

Hence, $D(f)$ is a countable union of closed sets in X.

Proof. By part 2 of Proposition 6.1.3, $a \in D(f)$ if and only if $\omega_f(a) > 0$, in which case there must be some positive integer n such that $\omega_f(a) \ge \frac{1}{n}$. Hence, (6.1.8) holds. Thus, $D(f)$ is a countable union of closed sets in X by part 1 of Proposition 6.1.3. $\square$

Corollary 6.1.5 says that there is no function $f : \mathbb{R} \to \mathbb{R}$ whose set of discontinuities is the irrationals; Example 6.1.10 shows that there is a function $f : \mathbb{R} \to \mathbb{R}$ whose set of discontinuities is the rationals.

Corollary 6.1.5. *Let Y be any metric space. Then there is no $f : \mathbb{R}^n \to Y$ with $D(f) = \not{\mathbb{Q}}^n$.*

Proof. Let $f : \mathbb{R}^n \to Y$. By Corollary 6.1.4, $D(f)$ is a countable union of closed sets in $\mathbb{R}^n$. Hence, if $D(f) = \not{\mathbb{Q}}^n$, then $\mathbb{Q}^n$ is a countable intersection of open sets in $\mathbb{R}^n$, contradicting Example 1.13.12. □

If X is a metric space with the discrete topology, then all maps f defined on X are continuous and no subset of X is a set of discontinuities for any function f. Theorem 6.1.7 gives conditions on a metric space that ensure that $D(f) \subset X$ is a countable union of closed sets in X. We will need the following lemma.

Lemma 6.1.6. *Let X be a metric space and assume that every open ball in X is uncountable. Let A be a closed set in X and denote by A^0 the set of all interior points of A. Let C be countable dense set in X. Then for each $x \in A$, there is some $r > 0$ such that if $0 < \epsilon < r$, the ball $B(x, \epsilon)$ contains u, v, with $v \in A \setminus (C \cap A^0)$ and*

$$u \in \begin{cases} C \cap A^0 & , \text{ if } x \in A^0 \\ X \setminus A & , \text{ if } x \in A \setminus A^0 \end{cases} \tag{6.1.9}$$

Proof. Assume first that $x \in A^0$. Then there is some $r > 0$ such that $B(x, \epsilon) \subset A$ for all $0 < \epsilon < r$. Let $0 < \epsilon < r$ be arbitrary. Since C is dense in X, there is some $u \in C \cap B(x, \epsilon)$. By Proposition 1.2.5, u is also an interior point of A. Hence, $u \in C \cap A^0$. By assumption, $B(x, \epsilon)$ is uncountable. Because C is countable, there must be some $v \in B(x, \epsilon)$ such that $v \notin C$. Thus, $v \in A \setminus (C \cap A^0)$.

Next, assume that $x \notin A^0$. Let $\epsilon > 0$ be arbitrary. Then, of course, $x \in B(x, \epsilon)$ and $x \in A \setminus (C \cap A^0)$. So, we can take $v = x$. Since $x \notin A^0$, the open ball $B(x, \epsilon)$ must contain some $u \in X \setminus A$. □

Theorem 6.1.7 (Sets of discontinuities). *Let X be a separable metric space such that every open ball in X is uncountable and let $F \subset X$. Then F is a countable union of closed sets in X if and only if there is some $f : X \to \mathbb{R}$ such that $D(f) = F$.*

In particular, since every open ball in a nonzero normed space is uncountable (see Exercise 2.4.9), the theorem holds if X is any nonzero separable normed space.

Example 6.1.8. Let X be any separable normed space. Then there is some dense set S of X and a function $f : X \to \mathbb{R}$ such that f is discontinuous at each point of S but continuous at every point of $X \setminus S$. To see this, let $S = \{x_1, x_2, \dots\}$ be a countable dense subset of X. Then S is a countable union of closed sets in X. Hence, by Theorem 6.1.7, such an f exists. $\triangle$

Example 6.1.9. In Theorem 6.1.7, it is important that every open ball in X be uncountable. If we set $X := \mathbb{Q}$ with the discrete metric, then the set $\mathbb{N}$ is a countable union of closed sets in X but there is no $f : X \to \mathbb{R}$ with $D(f) = \mathbb{N}$, since every $f : X \to \mathbb{R}$ is continuous on all of X. $\triangle$

Proof of Theorem 6.1.7. The "if" part was proved in Corollary 6.1.4. For the "only if" part, set $F := \bigcup_{k=1}^{\infty} F_k$, where each F_k is a closed set in X. We will show that there is some $f : X \to \mathbb{R}$ such that $D(f) = F$.

Set $A_1 := F_1$, $A_2 := A_1 \cup F_2$, $A_3 := A_2 \cup F_3$, and so on. Then (A_n) is a sequence of closed sets in X with $A_{n-1} \subset A_n$ and

$$F = \bigcup_{n=1}^{\infty} A_n. \tag{6.1.10}$$

Since X is separable, there is a countable dense set C in X. Set

$$f_n(x) := \begin{cases} 1 & , \text{ if } x \in A_n \setminus (C \cap A_n^0) \\ 0 & , \text{ otherwise.} \end{cases} \tag{6.1.11}$$

It is clear that $f_n = 0$ on the open set $X \setminus A_n$, so f_n is continuous at all points in $X \setminus A_n$. Let $x_0 \in A_n$. Apply Lemma 6.1.6 to conclude that for all $\epsilon > 0$ sufficiently small, there are u, v in $B(x_0, \epsilon)$ such that $u \in (X \setminus A_n) \cup (C \cap A_n^0)$ and $v \in A_n \setminus (C \cap A_n^0)$. Thus,

$$\Omega\big(f_n; B(x_0, \epsilon)\big) \geq f_n(v) - f_n(u) = 1. \tag{6.1.12}$$

Since (6.1.12) holds for all sufficiently small $\epsilon > 0$, we have $\omega_{f_n}(x_0) = 1$. Consequently, f_n is discontinuous at x_0. Since n and $x_0 \in A_n$ were arbitrary, we have thus shown that

$$D(f_n) = A_n \quad \text{for all } n \in \mathbb{N}. \tag{6.1.13}$$

Each f_n is an element of $B(X)$, the Banach space of all bounded functions on X, and $\|f_n\| = 1$. So

$$\sum_{n=1}^{\infty} \|4^{-n} f_n\| < \infty. \tag{6.1.14}$$

By the Weierstrass test for uniform convergence (Corollary 2.8.3), there is some $f \in B(X)$ such that

$$f = \sum_{n=1}^{\infty} 4^{-n} f_n \quad \text{uniformly on } X. \tag{6.1.15}$$