topological space and $\mathcal{A}$ is any sub-algebra of $C(X,\mathcal{K})$ having some of the same properties as the collection of polynomials on $[a,b]$.

Theorem 5.4.1 (The Stone-Weierstrass theorem). *Let X be a compact topological space and let $\mathcal{A} \subset C(X,\mathcal{K})$, where $\mathcal{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Assume that:*

1. *$\mathcal{A}$ is a sub-algebra of $C(X,\mathcal{K})$.*
2. *$\mathcal{A}$ vanishes at no points in X.*
3. *$\mathcal{A}$ separates points in X.*
4. *$\mathcal{A}$ is closed under complex conjugation.*

Then $\mathcal{A}$ is dense in $C(X,\mathcal{K})$, i.e., $\overline{\mathcal{A}} = C(X,\mathcal{K})$.

Of course assumption 4 above is automatically satisfied when $\mathcal{K} = \mathbb{R}$.

Thus if $\mathcal{A}$ satisfies these four algebraic conditions, then we can draw the sweeping topological conclusion that any continuous function $f : X \to \mathcal{K}$ can be uniformly approximated by elements of $\mathcal{A}$.

Proof. First consider the case when $\mathcal{K} = \mathbb{R}$. Thus, let $f \in C(X,\mathbb{R})$ and let $\epsilon > 0$. For any $x \in X$, Corollary 5.3.15 yields a $g_x \in \overline{\mathcal{A}}$ such that

$$g_x(x) = f(x) \text{ and } g_x(t) > f(t) - \epsilon \quad \text{for all } t \in X. \tag{5.4.1}$$

Put

$$V_x := \{\, t \in X \mid g_x(t) < f(t) + \epsilon \,\}. \tag{5.4.2}$$

Because g_x and f are continuous on X, it follows that V_x is open. Also, $x \in V_x$, since $g_x(x) = f(x) < f(x) + \epsilon$. Hence, $\{\, V_x \mid x \in X \,\}$ is an open cover for X and, by compactness, there are $x_1, \ldots, x_m$ in X such that

$$X = \bigcup_{k=1}^{m} V_{x_k}. \tag{5.4.3}$$

By Proposition 5.3.11, $\overline{\mathcal{A}}$ is a sub-algebra of $C(X,\mathbb{R})$. Let $h := \min_{1 \le k \le m} g_{x_k}$. Since each $g_{x_k} \in \overline{\mathcal{A}}$, Proposition 5.3.14 says that $h \in \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}}$. By (5.4.2) and (5.4.3),

$$h(t) < f(t) + \epsilon \quad \text{for all } t \in X. \tag{5.4.4}$$

Also, since $g_{x_k}(t) > f(t) - \epsilon$ for all $t \in X$ and all $k = 1, \ldots, n$, we must have $h(t) > f(t) - \epsilon$ for all $t \in X$. Hence, $\|h - f\| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we have shown that $f \in \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}}$.

Now assume that $\mathcal{K} = \mathbb{C}$. Consider the sub-algebra $\mathcal{A}_{\mathbb{R}}$ of $C(X,\mathbb{R})$ as in Proposition 5.3.17. Let $x \in X$. By assumption 4, $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $\mathcal{A}_{\mathbb{R}}$ for all $f \in \mathcal{A}$. By assumption 2, there is some $f \in \mathcal{A}$ such that $f(x) \neq 0$. Hence, either $(\operatorname{Re} f)(x) \neq 0$ or $(\operatorname{Im} f)(x) \neq 0$. Hence, $\mathcal{A}_{\mathbb{R}}$ vanishes at no points in X. Let a, b be distinct points in X. By assumption 3, there is some $g \in \mathcal{A}$ such that $g(a) \neq g(b)$. Since $g(a) \neq g(b)$, either $(\operatorname{Re} g)(a) \neq (\operatorname{Re} g)(b)$ or $(\operatorname{Im} g)(a) \neq (\operatorname{Im} g)(b)$. So, $\mathcal{A}_{\mathbb{R}}$ also separates points in X. Hence, $\mathcal{A}_{\mathbb{R}}$ satisfies all the assumptions in our theorem. Thus, by the first part just proved, $\mathcal{A}_{\mathbb{R}}$ is dense in $C(X, \mathbb{R})$. So, by Proposition 2.7.6, $\mathcal{D} := \{ u + iv \mid u, v \text{ in } \mathcal{A}_{\mathbb{R}} \}$ is dense in $C(X, \mathbb{C})$. Because $\mathcal{A}_{\mathbb{R}} \subset \mathcal{A}$ and $\mathcal{A}$ is an algebra over $\mathbb{C}$, we also have $\mathcal{D} \subset \mathcal{A}$. In particular, $\mathcal{A}$ must be dense in $C(X, \mathbb{C})$. □

Remark 5.4.2. Some treatments of the Stone-Weierstrass theorem replace assumption 2, "$\mathcal{A}$ vanishes at no points in X", by the stronger condition "$\mathcal{A}$ contains $\mathbf{1}$, the function that takes on the value 1 for all $x \in X$." This weakens the statement. Consider $\mathcal{A}_5$ in Example 5.3.8 – i.e., the collection of all functions of the form

$$S(x) := a_1 \cos x + \cdots + a_n \cos nx \text{ for } x \in [0, \pi], \tag{5.4.5}$$

for $a_1, \ldots, a_n \in \mathcal{K}$. Although $\mathcal{A}_5$ does not contain the constant function $\mathbf{1}$ on $[0, \pi]$, our version of the Stone-Weierstrass theorem and part 4 of Examples 5.3.8 tell us that $\overline{\mathcal{A}_5} = C([0, \pi], \mathcal{K})$. △

The next examples show that all four of the assumptions in Theorem 5.4.1 are needed.

Examples 5.4.3. 1. By part 2 of Examples 5.3.8, the collection $\mathcal{A}$ of real-valued functions on $[0, \pi]$ of the form $s(x) = a_0 + a_1 \sin x + \cdots + a_n \sin nx$ is not a sub-algebra of $C([0, \pi], \mathbb{R})$. Consequently, we may not apply Theorem 5.4.1 to conclude that $\overline{\mathcal{A}} = (C[0, \pi], \mathbb{R})$. In fact,

$$\overline{\mathcal{A}} \neq (C[0, \pi], \mathbb{R}). \tag{5.4.6}$$

This is because $s(0) = s(\pi)$ for all $s \in \mathcal{A}$, so that if $h \in (C[0, \pi], \mathbb{R})$ and $h(0) \neq h(\pi)$, then h cannot be a limit of sequence of functions in $\mathcal{A}$.

2. As in part 3 of Examples 5.3.8, set $f_n(x) := (x - 1)^n$ for $x \in X$. Let $\mathcal{A}_2 := \operatorname{span} \{ f_n \mid n \geq 1 \}$ in $C(X, \mathbb{R})$. Then $\mathcal{A}_2$ vanishes at 1. Thus any function $f \in C([0, 1], \mathbb{R})$ with $f(1) \neq 0$ cannot be in $\overline{\mathcal{A}_2}$.

3. Let $J := [-1, 1]$; as in part 3 of Example 5.3.8, let $\mathcal{A}_J$ be the collection of all $h : J \to \mathbb{R}$ of the form $h(x) := p(x^2)$, where $p \in \mathbb{R}[t]$ is arbitrary. We saw in that example that $\mathcal{A}_J$ does not separate points in J. Thus Theorem 5.4.1 cannot be applied to conclude that $\overline{\mathcal{A}_J} = C(J, \mathbb{R})$. In fact,

$$\overline{\mathcal{A}_J} \neq C(J, \mathbb{R}). \tag{5.4.7}$$

To see this, note that $f(-1) = f(1)$ for all $f \in \mathcal{A}_J$, so $h(-1) = h(1)$ for any $h \in \overline{\mathcal{A}_J}$. In particular, the identity function g defined by $g(x) := x$ is not in $\overline{\mathcal{A}_J}$.