Chapter 5

The Banach space $C(X)$

In this chapter we single out one of the most extensively studied Banach spaces, $C(X)$ – the Banach space of all continuous complex-valued functions on a *compact* topological space X. In the case $X = [a, b]$, we will give several important results, including theorems of Korovkin and Bernstein. These will lead us to the Weierstrass approximation theorem, which implies the separability of $C[a, b]$.

When X is a general compact topological space, we will study sub-collections of $C(X)$ that are closed under "multiplication" in $C(X)$; these collections are the "sub-algebras" in $C(X)$. This will lead to the Stone-Weierstrass theorem, a far-reaching generalization of the Weierstrass approximation theorem.

We will begin by defining *equicontinuity*, a concept needed for the *Arzela-Ascoli theorem*, which gives conditions for when a subset of $C(X)$ is compact.

5.1 The Arzela-Ascoli theorem

The result behind all of calculus is the statement that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded; this underlies such critical results as the mean value theorem, the uniform continuity of functions on compact sets (which makes Riemann sums converge and thus allows the definition of the integral), the fundamental theorem of algebra, and the spectral theorem for symmetric matrices.

In the function space setting of functional analysis, the analogous statement is the Arzela-Ascoli theorem, which says that if X is compact, a subset of $C(X)$ is compact if and only if it is closed, bounded, and *equicontinuous*.

Equicontinuity

Let X be a topological space and (Z, ρ) a metric space. Then $f : X \to Z$ is continuous if for each $x \in X$ and for all $\epsilon > 0$, there is some neighborhood U of

x such that

$$f(U) \subset B\big(f(x), \epsilon\big). \tag{5.1.1}$$

Let $S \subset C(X, Z)$, where as usual, $C(X, Z)$ denotes the collection of all continuous functions $X \to Z$. Then (5.1.1) holds for all $f \in S$, but in general, U depends on f. The case where we can choose a single U for all $f \in S$ warrants a special name.

Definition 5.1.1 (Equicontinuity). Let X be a topological space, Z a metric space, and $S \subset C(X, Z)$. Then S is *equicontinuous* if for each $x \in X$ and each $\epsilon > 0$, there is a neighborhood U of x such that *for all* $f \in S$,

$$f(U) \subset B_Z(f(x), \epsilon).$$

A sequence (f_n) in $C(X, Z)$ is equicontinuous if $\{f_n\}$ is equicontinuous.

Example 5.1.2. For each $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) := x^n$. Then (f_n) is not equicontinuous. Indeed, let U be any neighborhood U of 1 in $[0, 1]$. Find $\delta > 0$ such that $y := 1 - \delta \in U$. Then

$$|f_n(y) - f_n(1)| = 1 - (1 - \delta)^n > \frac{1}{2} \quad \text{for all } n \text{ sufficiently large.} \qquad \triangle$$

If X is a metric space, we can rewrite Definition 5.1.1: Let (X, d) and (Z, ρ) be metric spaces, and $S \subset C(X, Z)$. Then S is equicontinuous if for all $x \in X$ and each $\epsilon > 0$, there is some $\delta > 0$ such that for all $f \in S$ and all $y \in X$,

$$d(x, y) < \delta \implies \rho\big(f(x), f(y)\big) < \epsilon. \tag{5.1.2}$$

The metric on X lets us compare the distance between any two points in X, so in this setting we can also define *uniform equicontinuity*:

Definition 5.1.3 (Uniform equicontinuity). Let (X, d) and (Z, ρ) be metric spaces. Then $S \subset C(X, Z)$ is *uniformly equicontinuous* if for if each $\epsilon > 0$, there is some $\delta > 0$ such that for all $x, y \in X$ and all $f \in S$,

$$d(x, y) < \delta \implies \rho\big(f(x), f(y)\big) < \epsilon. \tag{5.1.3}$$

(This works because the metric allows us to choose a fixed δ that works for all points in X. Uniform equicontinuity cannot be defined when X is only a topological space, because in that setting we have no way to compare the sizes of neighborhoods around different points in X.)

Example 5.1.4 (Uniform equicontinuity). Let S be any collection of real-valued functions on $[a, b]$ and assume that there is some $M < \infty$ such that whenever $x \in (a, b)$, we have $|f'(x)| \leq M$ for all $f \in S$. Then S is uniformly equicontinuous. Indeed, given $\epsilon > 0$, set $\delta := \frac{\epsilon}{2M}$. Then for all $x, y \in [a, b]$ with $|x - y| < \delta$, we have (by the mean value theorem)

$$|f(x) - f(y)| \leq M\delta < \epsilon \quad \text{for all } f \in S. \qquad \triangle$$

Remark. This is a good occasion to consider the importance of the order of quantifiers. Let (X, d), (Z, ρ) be metric spaces and $\mathcal{F}$ is a collection of functions $X \to Z$. The four very different concepts below – continuity, uniform continuity, equicontinuity, and uniform equicontinuity – all differ only by the order of the quantifiers:

1. All elements of $\mathcal{F}$ are continuous, i.e., $\mathcal{F} \subset C(X, Z)$, if
$$\forall f \in \mathcal{F}, \quad \forall x \in X, \quad \forall \epsilon > 0, \quad \exists \delta > 0 \quad \text{such that} \quad \forall y \in X,$$
$d(x, y) < \delta \implies \rho\big(f(x), f(y)\big) < \epsilon$.
2. All $f \in \mathcal{F}$ are uniformly continuous if
$$\forall f \in \mathcal{F}, \quad \forall \epsilon > 0, \quad \exists \delta > 0 \quad \text{such that} \quad \forall x, y \in X,$$
$d(x, y) < \delta \implies \rho\big(f(x), f(y)\big) < \epsilon$.
3. $\mathcal{F} \subset C(X, Z)$ is equicontinuous if
$$\forall x \in X, \quad \forall \epsilon > 0, \quad \exists \delta > 0 \quad \text{such that} \quad \forall f \in \mathcal{F}, \quad \forall y \in X,$$
$d(x, y) < \delta \implies \rho\big(f(x), f(y)\big) < \epsilon$.
4. $\mathcal{F} \subset C(X, Z)$ is uniformly equicontinuous if
$$\forall \epsilon > 0, \quad \exists \delta > 0 \quad \text{such that} \quad \forall x, y \in X, \quad \forall f \in \mathcal{F},$$
$d(x, y) < \delta \implies \rho\big(f(x), f(y)\big) < \epsilon$.

Note that one can reverse the order of consecutive universal quantifiers and consecutive existential quantifiers, but changing the order of a $\forall$ and a $\exists$ changes the meaning. $\triangle$

The next result says that when X is a compact metric space, equicontinuity and uniform equicontinuity are equivalent.

Proposition 5.1.5. *Let $(X, d), (Z, \rho)$ be metric spaces. Assume that X is compact and $S \subset C(X, Z)$. Then the following are equivalent:*

1. *S is equicontinuous*
2. *S is uniformly equicontinuous.*

Proof. It is clear that if S is uniformly equicontinuous then it is equicontinuous. Conversely, assume S is equicontinuous. Then given any $\epsilon > 0$ and any $x \in X$, there is some $\eta_x > 0$ such that $f(B_X(x, \eta_x)) \subset B_Z(f(x), \epsilon)$ for all $f \in S$. Clearly, the collection $\mathcal{O} := \{ B_X(x, \eta_x) \mid x \in X \}$ is an open covering for X. Since X is compact, Theorem 1.11.7 implies that there is some $\delta > 0$ such that whenever $A \subset X$ and $\operatorname{diam} A < \delta$, then A is contained entirely in some element of $\mathcal{O}$. Thus, if $x, y \in X$ and $d(x, y) < \delta$, then $\rho(f(x), f(y)) < \epsilon$ for all $f \in S$. $\square$

Example 5.1.6 (Fredholm integral operator). Let $K : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and consider the Fredholm integral operator F_K of Section 3.2:

$$(F_K g)(y) := \int_a^b K(y,t)\, g(t)\, dt \quad \text{for all } y \in [a,b]. \tag{5.1.4}$$

Let $S \subset C([a,b],\mathbb{R})$ be bounded. We will show that $F_K(S)$ is a uniformly equicontinuous subset of $C([a,b],\mathbb{R})$. To see this, let $\epsilon > 0$ be given and denote the sup-norm on $C([a,b],\mathbb{R})$ by $\|\ \|$. Choose $C < \infty$ such that $\|g\| \le C$ for all $g \in S$. It was shown in (3.2.8) that there is some $\delta > 0$ such that whenever y_1, y_2 are in $[a,b]$ with $|y_1 - y_2| < \delta$, then for all $g \in S$,

$$|(F_K g)(y_1) - (F_K g)(y_2)| \le \epsilon \|g\|\,(b-a) \le \epsilon C(b-a). \tag{5.1.5}$$

Since $\epsilon > 0$ was arbitrary, $F_K(S)$ is uniformly equicontinuous. (We proved this directly, but since $[a,b]$ is compact, it would have been enough, by Proposition 5.1.5, to prove that $F_K(S)$ is equicontinuous. Note that we used the compactness of $[a,b]$ in the proof, when we referred to (3.2.8); that equation depends on the compactness of $[a,b] \times [c,d]$ to infer that K is uniformly continuous.) $\triangle$

Equicontinuity and uniform boundedness

Recall (Definition 1.13.18) the notions of pointwise and uniform boundedness for complex-valued functions defined on a set X. Exercise 5.1.19 asks you to show that for *linear* maps between normed spaces, uniform equicontinuity and uniform boundedness are equivalent. Proposition 5.1.8 asserts that if X is a compact topological space and $S \subset C(X)$ is pointwise bounded and equicontinuous, then S is uniformly bounded on X. It is our third uniform boundedness principle; the first two were Proposition 1.13.20 and Corollary 3.4.4 (the Banach-Steinhaus theorem). We will use the following lemma when we prove Proposition 5.1.8.

Lemma 5.1.7. *Let X be a compact topological space and let $S \subset C(X)$ be equicontinuous. Then for each $\epsilon > 0$, there are $x_1, \dots, x_n$ in X such that*

$$f(X) \subset \bigcup_{k=1}^{n} B(f(x_k), \epsilon) \quad \textit{for all } f \in S. \tag{5.1.6}$$

Proof. Let $\epsilon > 0$ be given. For each $x \in X$, there is a neighborhood U_x of x such that

$$f(U_x) \subset B(f(x), \epsilon) \quad \text{for all } f \in S. \tag{5.1.7}$$

The collection $\{U_x \,|\, x \in X\}$ is an open cover for X. Since X is compact, there exist $x_1, \dots, x_n$ in X such that $X = \bigcup_{k=1}^{n} U_{x_k}$. Hence, for all $f \in S$,

$$f(X) = \bigcup_{k=1}^{n} f(U_{x_k}) \subset \bigcup_{k=1}^{n} B\big(f(x_k), \epsilon\big) \qquad \square \tag{5.1.8}$$