Theorem 4.5.1 (Hilbert spaces). *Let $(X, \langle\, , \rangle)$ be an inner product space over $\mathcal{K}$. The following statements are equivalent:*

1. *X is a Hilbert space.*
2. *Every closed vector subspace M of X has the minimum distance property: for every $x \in X$, there is some $P_M x \in M$ such that*

$$d(x, M) = \|x - P_M x\|.$$

3. *X has the orthogonal decomposition property: if $M \subset X$ is a closed vector subspace, then M and $M^\perp$ are complementary subspaces in X.*
4. *X has the Riesz representation property: every continuous linear functional on X is g_z for some $z \in X$, where $g_z(x) := \langle x, z\rangle$ for all $x \in X$.*

Some of these implications have special names:

$1 \implies 2$ is the *minimum distance theorem.*
$1 \implies 3$ is usually called the *projection theorem.*
$1 \implies 4$ is the *Riesz representation theorem.*

Proof. $2 \implies 3$: *The minimum distance property implies the orthogonal decomposition property.* This is Proposition 4.4.12.

$3 \implies 4$: *The orthogonal decomposition property implies the Riesz representation property.* This is Proposition 4.4.14

$4 \implies 1$: *If X has the Riesz representation property, X is a Hilbert space.* We will see that if X is not a Hilbert space, then there is some continuous linear functional on X that is not of the form g_z for any $z \in X$.

So assume that $(X, \langle\ ,\ \rangle)$ is not a Hilbert space. By part 1 of Proposition 4.3.5, there is a Hilbert space completion $(H, \langle\ ,\ \rangle')$ of X. Hence, there is a proper dense vector subspace Y of H and a unitary isomorphism $T : X \to Y$.

Since $Y \neq H$, we can choose some $h_0 \in H$ with $h_0 \notin Y$. Define $f : Y \to \mathcal{K}$ by $f(y) := \langle y, h_0\rangle'$ for all $y \in Y$. Then f is a continuous linear functional on Y.

The remainder of the proof consists of two parts:

(a) We will first verify that Y does not have the Riesz representation property, by showing that there is no $u \in Y$ for which

$$f(y) = \langle y, u\rangle' \quad \text{for all } y \in Y. \tag{4.5.1}$$

This is the heart of the proof.

(b) Then using the fact that X and Y are unitarily isomorphic together with part (a), we will show that X too does not have the Riesz representation property.

Part (a) If there were such a $u \in Y$, then we would have

$$\langle y, h_0 - u \rangle' = 0 \quad \text{for all } y \in Y. \tag{4.5.2}$$

But then, by continuity of inner products, we must have

$$\langle y, h_0 - u \rangle' = 0 \quad \text{for all } y \in \overline{Y} = H, \tag{4.5.3}$$

and this would imply that $h_0 - u = \mathbf{0}$, a contradiction since $h_0 \notin Y$.
Part (b) Now define $\eta : X \to \mathcal{K}$ by

$$\eta(x) := f(Tx). \tag{4.5.4}$$

We will show that η is a continuous linear functional on X and $\eta \neq g_z$ for any $z \in X$. Since both f and T are continuous, η is a continuous linear functional on X. If $\eta = g_z$ for some $z \in X$, then $g_z = f \circ T$, so that

$$g_z \circ T^{-1} = f. \tag{4.5.5}$$

Because T is a unitary isomorphism, so is T^{-1}, and (4.5.5) would imply that

$$\begin{aligned} f(y) = \langle T^{-1}(y), z \rangle &= \left\langle T^{-1}y, T^{-1}(Tz) \right\rangle \\ &= \langle y, Tz \rangle' \quad \text{for all } y \in Y. \end{aligned} \tag{4.5.6}$$

Since $Tz \in Y$, this contradicts part (a). Hence, $\eta \neq g_z$ for any $z \in X$. Consequently, (4) implies (1).

$1 \Longrightarrow 2$: *Every closed vector subspace of a Hilbert space has the minimum distance property.*

This proof uses the parallelogram law. Let $M \subset X$ be a closed vector subspace and $y_0 \in X$. Put $d := d(y_0, M)$. Choose $(x_n) \subset M$ with $\|x_n - y_0\| \to d$. If (x_n) is Cauchy, then since X is complete, there is some $m_0 \in \overline{M} = M$ such that $x_n \to m_0$, and thus, by the continuity of the norm,

$$\|m_0 - y_0\| = \left\| \lim_{n\to\infty} x_n - y_0 \right\| = \lim_{n\to\infty} \|x_n - y_0\| = d. \tag{4.5.7}$$

Thus, we will show that (x_n) is Cauchy. Since M is a vector subspace, $\frac{x_m + x_n}{2}$ is in M. Thus, $\|y_0 - \frac{x_m+x_n}{2}\| \geq d$. Hence, by the parallelogram law,

$$\begin{aligned} \|x_n - x_m\|^2 &= \|(y_0 - x_m) - (y_0 - x_n)\|^2 \\ &= 2\left(\|y_0 - x_m\|^2 + \|y_0 - x_n\|^2\right) - \|(y_0 - x_m) + (y_0 - x_n)\|^2 \\ &= 2\left(\|y_0 - x_m\|^2 + \|y_0 - x_n\|^2\right) - 4\left\|y_0 - \frac{x_m + x_n}{2}\right\|^2 \\ &\leq 2\left(\|y_0 - x_m\|^2 + \|y_0 - x_n\|^2\right) - 4d^2. \end{aligned} \tag{4.5.8}$$

Hence, $\|x_n - x_m\| \to 2(d^2 + d^2) - 4d^2 = 0$ as $n, m \to \infty$, so (x_n) is Cauchy. Consequently, M has the minimum distance property. $\square$