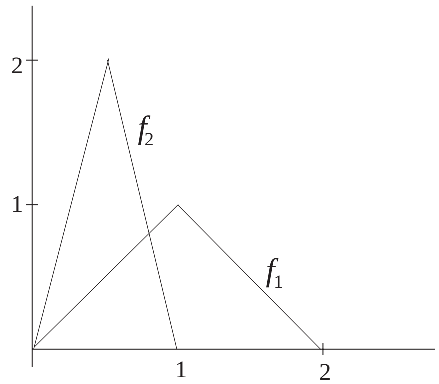

Figure 3.2: For a given n, let f_n be the function from $[0, \infty)$ to $\mathbb{R}$ that equals 0 at 0 and on $[2/n, \infty)$, that equals n at $1/n$, and whose graph between 0 and $1/n$ and between $1/n$ and $2/n$ consists of the straight lines connecting those points, as shown above for $n = 1$ and $n = 2$. Then $\sup_n |f_n(x)|$ is finite for every x; in fact, $\lim_{n\to\infty} f_n(x) = 0$. In the terminology of Theorem 3.4.1, $D = [0, \infty)$, so in particular, D is fat in $[0, \infty)$. Yet $\lim_{n\to\infty}(\sup_{x\in[0,\infty)} |f_n(x)|) = \infty$. This does not contradict Theorem 3.4.1, because the f_n are not linear. Recall that we already saw these f_n in Figure 1.1 illustrating uniform convergence.

The Banach-Steinhaus theorem

The Banach-Steinhaus theorem (Corollary 3.4.4) says that under appropriate conditions, a collection of bounded linear operators that is pointwise bounded is uniformly bounded. It follows from the theorem below. When reading Theorem 3.4.1, keep in mind the sequence of functions f_n of Figure 3.2: the theorem is not at all obvious.

Theorem 3.4.1. *Let X, Y be normed spaces and let $\mathcal{S} \subset \mathcal{B}(X, Y)$. Assume that $D := \{ x \in X \mid \sup_{T\in\mathcal{S}} \|Tx\| < \infty \}$ is fat in X. Then*

$$\sup_{T\in\mathcal{S}} \|T\| < \infty. \tag{3.4.1}$$

In particular, $D = X$.

Although $\|T\| < \infty$ for any $T \in \mathcal{B}(X, Y)$, the sup of the $\|T\|$ is not necessarily bounded. For instance, let I be the identity map on a nonzero normed space X and let $\mathcal{S} := \{ nI \mid n \in \mathbb{N} \}$. Then $\mathcal{S} \subset \mathcal{B}(X)$, but $\sup_{T\in\mathcal{S}} \|T\| = \infty$. Here, $D = \{\mathbf{0}\}$.

Proof of Theorem 3.4.1. Denote the norms on both X and Y by $\| \ \|$. Since each $T \in \mathcal{S}$ is continuous, the inverse image by T of a closed set is closed, and so, by Corollary 1.2.12, $E := \bigcap_{T\in\mathcal{S}} T^{-1}(\frac{1}{2}\overline{B_Y})$ is closed in X. Let $x \in D$. Then there is some integer n such that $Tx \in nB_Y = 2n(\frac{1}{2}B_Y) \subset 2n(\frac{1}{2}\overline{B_Y})$

for all $T \in \mathcal{S}$. Hence, $D \subset \bigcup_{n=1}^{\infty} nE$. Since D is fat in X, some nE must also be fat in X by part 4 of Proposition 1.13.4. Hence, E is also fat in X by Example 2.5.12. Because E is fat and closed, E must have an interior point in X (see Definition 1.13.1). Let x_0 be any interior point of E. Then $-x_0 + E$ is a neighborhood of $\mathbf{0}$ in X. Hence, there is some $\delta > 0$ such that $\delta B_X \subset -x_0 + E$. Let $T \in \mathcal{S}$. Then $T(E) \subset \frac{1}{2}\overline{B_Y}$, and, since $x_0 \in E$, we have $Tx_0 \in \frac{1}{2}\overline{B_Y}$, so that $T(\delta B_X) \subset -Tx_0 + T(E) \subset \frac{1}{2}\overline{B_Y} - \frac{1}{2}\overline{B_Y} \subset \overline{B_Y}$. Hence $T(B_X) \subset \frac{1}{\delta}\overline{B_Y}$, so

$$T(\overline{B_X}) \subset T(2B_X) \subset \frac{2}{\delta}\,\overline{B_Y}. \tag{3.4.2}$$

Since $T \in \mathcal{S}$ was arbitrary, $\sup_{T\in\mathcal{S}} \|T\| \le \frac{2}{\delta}$, proving (3.4.1).

To see that $D = X$, note that for any $x \in X$, we have $\|T(\frac{x}{\|x\|+1})\| \le \frac{2}{\delta}$, i.e., $\|Tx\| < \frac{2}{\delta}(\|x\| + 1)$ for all $T \in \mathcal{S}$. So $x \in X \implies x \in D$.

Example 3.4.2. Recall (Proposition 2.9.6) that $(\mathbf{e}_k)$ is a Schauder basis for l^2. Thus, each $\mathbf{x} \in l^2$ can be written uniquely as $\mathbf{x} = \sum_{k=1}^{\infty} x_k\,\mathbf{e}_k$. For $n \in \mathbb{N}$, define $T_n\mathbf{x} := n\,x_n\,\mathbf{e}_n$. Then each T_n is linear and

$$\|T_n\mathbf{x}\| = |n\,x_n| \le n\,\|\mathbf{x}\|. \tag{3.4.3}$$

Thus, $\|T_n\| \le n$. Since $\|T_n\mathbf{e}_n\| = n$, we have $\|T_n\| = n$ for all n, by Corollary 3.1.13. Hence, $\sup_{n\in\mathbb{N}} \|T_n\| = \infty$. By Theorem 3.4.1,

$$\left\{ \mathbf{x} \;\middle|\; \sup_n \|T_n(\mathbf{x})\| < \infty \right\} \tag{3.4.4}$$

is thin in l^2. △

Example 3.4.3. Let $X := \operatorname{span}\{\mathbf{e}_k\}$ with the induced norm from l^2. Define $T_n \in \mathcal{B}(X)$ by

$$T_n(\mathbf{e}_i) := \begin{cases} \mathbf{0} & , \text{ if } i \ne n \\ n\mathbf{e}_i & , \text{ if } i = n. \end{cases} \tag{3.4.5}$$

Let $D := \left\{ x \in X \;\middle|\; \sup_{n\ge1} \|T_n x\| < \infty \right\}$. Then $D = X$, but $\|T_n\| = n$ for all n. This does not contradict Theorem 3.4.1, since X is thin in itself by Example 3.3.10. △

Since every Banach space is fat in itself (Theorem 1.13.6), then when X is Banach, Theorem 3.4.1 implies the following. It is our second uniform boundedness principle.

Corollary 3.4.4 (Banach-Steinhaus theorem). *Let X, Y be normed spaces over the same field $\mathcal{K}$. Let $\mathcal{S} \subset \mathcal{B}(X, Y)$. Assume that*

1. *X is a Banach space.*
2. *$\sup_{T\in\mathcal{S}} \|Tx\| < \infty$ for all $x \in X$.*

Then there is some $C < \infty$ such that

$$\|T\| \le C \text{ for all } T \in \mathcal{S}. \tag{3.4.6}$$