Chapter 3

Operators on normed spaces

In this chapter we investigate continuous functions from one normed space to another. The class of all such functions is so large that any attempt to understand their properties will fail, so we will focus on those continuous functions that interact with the vector space structure in a meaningful way. These are the continuous *linear* operators between normed spaces. Thus, results obtained for continuous functions between metric spaces and whatever knowledge we have from linear algebra can come into play.

3.1 Continuous linear maps

Recall (Definition 1.5.4) that a map $T : X \to Y$ between topological spaces is continuous at $x_0 \in X$ if for every neighborhood V of $T(x_0)$ there is a neighborhood U of x_0 such that $T(U) \subset V$. The map T is continuous on X if it is continuous at each $x \in X$.

However, if T is *linear*, then if T is continuous at a single point, it is continuous on *all* of X.

Proposition 3.1.1. *Let X, Y be topological vector spaces over the same field and assume that $T : X \to Y$ is linear. Then the following are equivalent:*

1. *T is continuous at $\mathbf{0}$.*
2. *T is continuous on X.*
3. *T is continuous at some point of X.*

If Y is a normed space, then each of the above is equivalent to

4. *$T(U)$ is a bounded subset of Y for some neighborhood U of $\mathbf{0}$.*

Proof. $1 \implies 2$: Let $x_0 \in X$; let V be any neighborhood of Tx_0. Then $-Tx_0 + V$ is a neighborhood of $\mathbf{0} = T\mathbf{0}$. By the assumed continuity of T at $\mathbf{0}$, there is a neighborhood U of $\mathbf{0}$ such that $T(U) \subset -Tx_0 + V$. So, by the linearity of T, we have $T(x_0 + U) \subset V$. By Theorem 2.5.11, $x_0 + U$ is a neighborhood of x_0. Hence, T is continuous at x_0.

$2 \Longrightarrow 3:$ This is obvious.

$3 \Longrightarrow 1:$ Assume T is continuous at $x_0 \in X$. Let V be any neighborhood of $T\mathbf{0} = \mathbf{0}$. Then, by Theorem 2.5.11, $Tx_0 + V$ is a neighborhood of Tx_0 so that, by the continuity of T at x_0, there is a neighborhood U of x_0 such that $T(U) \subset Tx_0 + V$. But then $T(-x_0 + U) = -Tx_0 + T(U) \subset V$ and $-x_0 + U$ is a neighborhood of $\mathbf{0}$ in X. Hence, T is continuous at $\mathbf{0}$.

Now assume that Y is a normed space and let $\| \; \|$ denote the norm on Y.

$4 \Longrightarrow 1:$ Assume (4). Then there is a neighborhood U of $\mathbf{0}$ in X and a constant $M > 0$ such that $\|Tu\| < M$ for all $u \in U$. Let $\epsilon > 0$ be arbitrary. Then by Theorem 2.5.11, $\frac{\epsilon}{M}U$ is a neighborhood of $\mathbf{0}$ in X and $T\left(\frac{\epsilon}{M}U\right) \subset B_Y(\mathbf{0}, \epsilon)$.

$1 \Longrightarrow 4:$ Choose a neighborhood U of $\mathbf{0}$ in X such that $T(U) \subset B_Y(\mathbf{0}, 1)$. Let W be any neighborhood of $\mathbf{0}$ in Y. Then $B_Y(\mathbf{0}, 1) \subset tW$ for all t sufficiently large, and hence $T(U) \subset tW$ for all such t. Thus, $T(U)$ is bounded. □

Bounded linear operators

Now we will show that a linear map between *normed spaces* is continuous if and only if it is bounded in the sense defined below.

Definition 3.1.2 (Bounded linear operator). Let $(X, \| \; \|_1), (Y, \| \; \|_2)$ be normed spaces over the same field $\mathcal{K}$. A linear map $T : X \to Y$ is a *bounded linear operator* if there is a positive constant M satisfying

$$\|Tx\|_2 \leq M\|x\|_1 \text{ for all } x \in X. \tag{3.1.1}$$

We will denote by $\mathcal{B}(X, Y)$ the set of bounded linear operators from X to Y, and by $\mathcal{B}(X)$ the set of bounded linear operators from X to X. (Note that this notation is not analogous to $C(X)$, which denotes continuous functions from X to $\mathbb{C}$, not to X.)

Remark. In calculus, a function $f : S \to \mathbb{R}$ is “bounded” if there is a fixed constant C such that $|f(x)| \leq C$ for all $x \in S$. This definition is useless for linear operators, since *the only linear operator $T : X \to Y$ between normed spaces X and Y for which there is a constant C satisfying $\|Tx\| \leq C$ for all $x \in X$ is the zero operator.*

Indeed, let $T : X \to Y$ be a nonzero linear function. Then there is some $x_0 \in X$ such that $Tx_0 \neq \mathbf{0}$. Since T is linear, $x_0 \neq \mathbf{0}$. For all $\alpha \in \mathcal{K}$, we have $\|T(\alpha x_0)\| = |\alpha| \; \|Tx_0\|$, so $\|T(\alpha x_0)\|$ can be made as large as we like by taking $|\alpha|$ large enough. Consequently, if a linear map $T : X \to Y$ satisfies $\|Tx\| \leq C < \infty$, then $T = \mathbf{0}$. △

The next result says that for linear maps between normed spaces, *boundedness and continuity are equivalent.* Thus, a linear operator $T : X \to Y$ between normed spaces is continuous if and only if each vector $x \in X$ is not stretched too much by T.

Theorem 3.1.3 (Bounded linear operators). *Let X, Y be normed spaces and let $T : X \to Y$ be linear. Let B be the closed unit ball in Y. The following statements are equivalent:*

1. *T is bounded.*
2. *T is uniformly continuous on X.*
3. *T is continuous on X.*
4. *T maps bounded subsets of X onto bounded subsets of Y.*
5. *The interior of $T^{-1}(B)$ is not empty.*

PROOF. Denote the norm on X by $\| \; \|_1$ and the norm on Y by $\| \; \|_2$.

$1 \Longrightarrow 2$: If T is bounded, there is some $C < \infty$ such that $\|Tx\|_2 \leq C\|x\|_1$ for all $x \in X$. Thus,

$$\|Tx - Ty\|_2 = \|T(x-y)\|_2 \leq C\|x-y\|_1, \tag{3.1.2}$$

so that T is uniformly continuous.

$2 \Longrightarrow 3$: This is obvious.

$3 \Longrightarrow 4$: If T is continuous on X, then, by Proposition 3.1.1, there is a neighborhood U of $\mathbf{0}$ and a constant C such that $\|Tu\|_2 \leq C$ for all $u \in U$. Choose $r > 0$ such that $B_r := B(\mathbf{0}, r) \subset U$. Let A be any bounded subset of X. Then there is some positive constant λ such that $A \subset \lambda B_r$. Thus, $T(A) \subset \lambda T(B_r) \subset \lambda T(U)$, so that $\|Ta\| \leq \lambda C$ for all $a \in A$. Hence, $T(A)$ is bounded in Y.

$4 \Longrightarrow 1$: Assume that T maps bounded subsets of X onto bounded subsets of Y. Then the image of the unit sphere $\{ x \in X \mid \|x\|_1 = 1 \}$ under T is a bounded subset of Y. Thus there is some constant M such that $\|Tx\|_2 \leq M$ whenever $\|x\|_1 = 1$. Hence, for $z \in X$ with $z \neq \mathbf{0}$, we have $T(\frac{z}{\|z\|_1}) \leq M$, i.e., $\|Tz\|_2 \leq M\|z\|_1$.

$3 \Longrightarrow 5$: Assume that T is continuous on X. Since B contains a nonempty open set, $T^{-1}(B)$ must contain a nonempty open set (see Theorem 1.5.1).

$5 \Longrightarrow 3$: Assume the interior of $T^{-1}(B)$ is nonempty. Then there is some $x_0 \in X$ and some $r > 0$ such that $B(x_0, r) \subset T^{-1}(B)$, so $T(B(x_0, r)) \subset B$, so

$$T(B(\mathbf{0}, r)) = T(-x_0 + B(x_0, r)) \subset -Tx_0 + B. \tag{3.1.3}$$

Since $-Tx_0 + B$ is a bounded subset of Y, Proposition 3.1.1 implies that T is continuous on X. $\square$

Remarks. Theorem 3.1.3 is not true if we do not require T to be linear:

1. There are many nonlinear maps that satisfy (3.1.1) but are not continuous. For example, let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) := 1$ if $|x| > 1$ and by

$T(x) := x$ otherwise. Then T is not linear. Clearly, T is bounded, since $|T(x)| \leq |x|$ for all $x \in \mathbb{R}$, but T is not continuous on $\mathbb{R}$.

2. There are continuous nonlinear maps that do not satisfy (3.1.1). Let $S : \mathbb{R} \to \mathbb{R}$ be defined by $S(x) := x^2$. Then S is continuous on $\mathbb{R}$, but there is no constant M for which $x^2 = |S(x)| \leq M|x|$ for all $x \in \mathbb{R}$.
3. There are functions f that are not continuous but such that $f^{-1}(B)$ has nonempty interior. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := x$ if $x \in [-1, -1]$ and $f(x) := 5$ if $x \notin [-1, 1]$. Let $B := [-1, 1]$, the closed unit ball in $\mathbb{R}$ centered at 0. Then $f^{-1}(B)$ has nonempty interior because it contains the open interval $(-1, 1)$. But clearly f is not continuous. △

Operator norms

A *linear* operator, bounded or not, has a norm.

Definition 3.1.4 (Operator norm). Let X, Y be normed spaces over the same field and let $T : X \to Y$ be any linear operator. The *operator norm* of T, denoted by $\|T\|$, is

$$\|T\| := \sup \{ \|Tx\| \mid x \in X, \|x\| \leq 1 \} . \tag{3.1.4}$$

Note that $\|T\|$ may be infinite. Note also that $\| \ \|$ in Definition 3.1.4 denotes three different norms: $\|T\|$ is the operator norm, $\|x\|$ is the norm on X, and $\|Tx\|$ is the norm on Y.

Proposition 3.1.5 (Other formulas for the operator norm). *Let X, Y be normed spaces over the same field and let $T : X \to Y$ be a linear operator. Assume that $X \neq \{\mathbf{0}\}$. Put*

$$A := \sup \{ \|Tx\| \mid x \in X, \|x\| = 1 \} , \tag{3.1.5}$$

$$B := \sup \left\{ \frac{\|Tx\|}{\|x\|} \mid x \in X, x \neq \mathbf{0} \right\} . \tag{3.1.6}$$

Then $\|T\| = A = B$.

Proof. Assume that $X \neq \{\mathbf{0}\}$. It's clear that $A \leq \|T\|$. Let $x \in X$ and $x \neq \mathbf{0}$. Then $z := \frac{x}{\|x\|}$ has norm 1, so that

$$\frac{\|Tx\|}{\|x\|} = \|Tz\| \leq A. \tag{3.1.7}$$

This shows that $B \leq A$. Conversely, if $\|x\| = 1$, then $\|Tx\| = \frac{\|Tx\|}{\|x\|} \leq B$, so that $A \leq B$. Hence, $A = B$. Finally, (3.1.7) shows that $\|Tx\| \leq A\|x\| \leq A$ if $\|x\| \leq 1$. Thus, $\|T\| \leq A$. So $\|T\| = A$. □