2.7 Banach spaces

Recall that a metric space X is *complete* if every Cauchy sequence in X converges to a point in X. Normed spaces whose induced metric spaces are complete are given a special name.

Definition 2.7.1 (Banach space). A *Banach space* is a normed space whose induced metric space is complete.

Theorem 2.7.2 (Banach spaces).

1. *The following normed spaces (with the norms defined in Section 2.3) are all Banach spaces:*

$$\mathbb{R}^n, \quad \mathbb{C}^n, \quad C[a,b], \quad l^p, \quad l^\infty.$$

2. *A closed vector subspace of a Banach space is itself a Banach space.*

Proof. 1. We saw in Section 2.3 that the metric spaces induced by the normed spaces $\mathbb{R}^n$, $\mathbb{C}^n$, $C[a,b]$, l^p, and l^∞ are the metric spaces introduced in Chapter 1. By Example 1.8.4 and Theorem 1.8.5, these metric spaces are complete.

2. This follows from Proposition 1.8.8 and the fact (Section 2.3) that the metric induced by the norm on M is the induced metric on X restricted to M.

Example 2.7.3. We saw in Example 2.4.6 that the set c of all convergent sequences in $\mathbb{C}$ is a closed set of l^∞. Clearly, c is also a vector subspace of l^∞. So c is a Banach space when considered as a normed subspace of l^∞. △

Example 2.7.4 (The Banach space $C(X,Y)$). Let X be a compact topological space and let $(Y, \|\;\|')$ be a Banach space over $\mathcal{K}$. As in Corollary 1.11.18, let $C(X,Y) := \{\, f : X \to Y \mid f \text{ is continuous} \,\}$. For $f, g \in C(X,Y)$, $\alpha \in \mathcal{K}$, and all $x \in X$, define $(f+g)(x) := f(x) + g(x)$ and $(\alpha f)(x) := \alpha f(x)$ (on the right side of these equations, we use the vector space operations in Y). Then $f+g$ and αf are also in $C(X,Y)$, so $C(X,Y)$ is a vector space over $\mathcal{K}$.

For $f \in C(X,Y)$, the function $x \mapsto \|f(x)\|'$ is a continuous function on the compact set X, since it is the composition of $f : X \to Y$ and $\|\;\|' : Y \to \mathbb{R}$, which are both continuous. Hence $x \mapsto \|f(x)\|'$ is bounded and achieves its supremum. Thus we may give $C(X,Y)$ the sup-norm

$$\|f\| := \sup_{x \in X} \|f(x)\|' \quad \text{for all } f \in C(X,Y). \tag{2.7.1}$$

Then it is easy to verify that $\|\;\|$ is a norm on $C(X,Y)$. By Corollary 1.11.18, $(C(X,Y), \|\;\|)$ is a Banach space over $\mathcal{K}$. △

Example 2.7.5 (The Banach space $C(X,\mathcal{K}^n)$). A special case of $C(X,Y)$ is the space $C(X,\mathcal{K}^n)$ of continuous vector-valued functions. Every function

$\mathbf{f} : X \to \mathcal{K}^n$ can be written

$$\mathbf{f}(x) := \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \tag{2.7.2}$$

where $f_1, \ldots, f_n$ are unique functions from X to $\mathcal{K}$. Note that $\mathbf{f}$ is in $C(X, \mathcal{K}^n)$ if and only if each f_k is in $C(X, \mathcal{K})$. This can be proved using the fact that for each k, the map $\mathcal{K}^n \to \mathcal{K}$ taking each $\mathbf{x}$ to x_k is continuous. △

Proposition 2.7.6. *Let X be a compact topological space. Assume that $C(X, \mathbb{R}^n)$ has a dense subset $\mathcal{D}_0$. Define*

$$\mathcal{D} := \mathcal{D}_0 + i\mathcal{D}_0 = \{\, \mathbf{g} + i\mathbf{h} \mid \mathbf{g}, \mathbf{h} \in \mathcal{D}_0 \,\}. \tag{2.7.3}$$

Then $\mathcal{D}$ is a dense subset of $C(X, \mathbb{C}^n)$.

PROOF. Let $\mathbf{f} \in C(X, \mathbb{C}^n)$ and $\epsilon > 0$ be given. Write $\mathbf{f} = \mathbf{g} + i\mathbf{h}$ for $\mathbf{g}, \mathbf{h} \in C(X, \mathbb{R}^n)$. By hypothesis, there exist functions $\mathbf{g}_1, \mathbf{h}_1 \in \mathcal{D}_0$ satisfying $\|\mathbf{g} - \mathbf{g}_1\| < \epsilon/2$ and $\|\mathbf{h} - \mathbf{h}_1\| < \epsilon/2$. Then $\mathcal{D}$ is dense in $C(X, \mathbb{C}^n)$, since

$$\|\mathbf{f} - (\mathbf{g}_1 + i\mathbf{h}_1)\| = \|\mathbf{g} + i\mathbf{h} - (\mathbf{g}_1 + i\mathbf{h}_1)\| \le \|\mathbf{g} - \mathbf{g}_1\| + \|i\mathbf{h} - i\mathbf{h}_1\| < \epsilon. \qquad \square$$

Example 2.7.7 ($C^1[-1,1]$ with the sup-norm is not a Banach space). Denote by $C^1[-1,1]$ the collection of $f \in C([-1,1], \mathbb{R})$ with derivative f' continuous on $[-1,1]$; at the endpoints -1 and 1, we mean the one-sided derivatives. Then $C^1[-1,1]$ is a vector subspace of $C([-1,1], \mathbb{R})$, and thus a normed space, under the sup-norm on $C([-1,1], \mathbb{R})$. For $n \in \mathbb{N}$, define $f_n : [-1,1] \to \mathbb{R}$ by

$$f_n(x) := |x|^{1+\frac{1}{n}}. \tag{2.7.4}$$

Then $f_n'(0) = 0$ for all n. It follows that (f_n) is a sequence in $C^1[-1,1]$. Let $g(x) := |x|$ for all $x \in [-1,1]$. It is easy to verify that $f_n(x) \le f_{n+1}(x)$ for all $x \in [-1,1]$. Thus (f_n) is an increasing sequence of functions converging pointwise on the compact interval $[-1,1]$ to the continuous function g:

$$\lim_{n\to\infty} f_n(x) = g(x) \quad \text{for all } x \in [-1,1]. \tag{2.7.5}$$

Hence, by Dini's theorem (Proposition 1.11.22), (f_n) converges uniformly on $[-1,1]$ to g. Recall from (1.7.25) and Definition 1.7.20 that convergence with respect to the sup-norm is the same as uniform convergence. Thus (f_n) converges in $C^1[-1,1]$ with the sup-norm. Since (f_n) is a Cauchy sequence in $C^1[-1,1]$ that converges to $g \notin C^1[-1,1]$, we conclude that $C^1[-1,1]$ with the sup-norm is not a Banach space.