Chapter 2

Normed spaces and topological vector spaces

Functional analysis is mainly an attempt to do linear algebra in infinite-dimensional vector spaces. This requires generalizing such basic notions as linear independence, span, dimension, basis, linear transformation, eigenvectors, and eigenvalues.

In particular, we will want to be able to express a vector x in an infinite-dimensional vector space as the infinite series $x = \sum_{k=1}^{\infty} c_k v_k$, where the c_k are scalars and the v_k are basis vectors. For such an expression to make sense, the infinite sum must converge. But to speak of convergence we need to be able to measure "distance". In this chapter we introduce *normed spaces* – spaces that are both metric spaces and vector spaces, where the vector space structure interacts with the metric in a natural and meaningful way.

But first, in Section 2.1, we will give some examples of linear maps defined on infinite-dimensional vector spaces. In Section 2.2 we will discuss *Hamel bases*, which don't require the structure of a norm.

2.1 Linear operators on function spaces

We will assume that you are familiar with finite-dimensional vector spaces, which are all essentially the same as $\mathbb{C}^n$ (more precisely, any n-dimensional vector space over $\mathbb{C}$ is isomorphic to $\mathbb{C}^n$). For a brief review, see Appendix B. As stated in Theorem B.5.2, every linear transformation $T : \mathbb{C}^n \to \mathbb{C}^m$ is given by multiplication by the $m \times n$ matrix $[T]$:

$$T(\mathbf{v}) = [T]\mathbf{v}, \tag{2.1.1}$$

where the ith column of $[T]$ is $T(\mathbf{e}_i)$. Moreover, using the "abstract to concrete function" discussed in Appendix B.5, a linear transformation between finite-dimensional abstract vector spaces can be translated into a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$. Thus in finite dimensions, all linear transformations can be understood in terms of matrix multiplication.

Linear functional analysis is largely concerned with linear transformations between *infinite-dimensional* vector spaces. Often, but not always, the elements of these vector spaces are functions; in that case the spaces are called, naturally enough, *function spaces*.

In greatest generality, one can consider the collection $\mathcal{F}(X, W)$ of all functions $f : X \to W$ where X and W are arbitrary nonempty sets. When W is a vector space over $\mathcal{K}$, the collection $\mathcal{F}(X, W)$ can be turned into a vector space over $\mathcal{K}$ in a natural way: if f, g are in $\mathcal{F}(X, W)$ and $\alpha \in \mathcal{K}$, define addition and scalar multiplication by

$$(f + g)(x) := f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) := \alpha f(x). \tag{2.1.2}$$

We call these operations *pointwise addition* and *pointwise multiplication*. Then $\mathcal{F}(X, W)$ is a vector space over $\mathcal{K}$.

However, if X is infinite, then $\mathcal{F}(X, W)$ is too big for us to say anything interesting about it; functions in $\mathcal{F}(X, W)$ can be too wild. In practice, we will be interested in vector subspaces of $\mathcal{F}(X, \mathbb{C})$. In particular, we will be interested in *linear* maps that can be defined on such subspaces.

Recall that if X, Y are vector spaces over the same field $\mathcal{K}$, then a function $f : X \to Y$ is linear if for all x, y in X and all $\alpha \in \mathcal{K}$,

$$f(x + y) = f(x) + f(y) \quad \text{and} \quad f(\alpha x) = \alpha f(x). \tag{2.1.3}$$

Here are three important examples of linear maps defined on subspaces of $\mathcal{F}(X, W)$.

Example 2.1.1 (Integration). The collection $\mathcal{R}[a, b]$ of all real-valued Riemann integrable functions on a bounded, closed interval $[a, b] \subset \mathbb{R}$ is a vector subspace of $\mathcal{F}([a, b], \mathbb{R})$. In particular, $\mathcal{R}[a, b]$ is a vector space over $\mathbb{R}$.

Define $T : \mathcal{R}[a, b] \to \mathbb{R}$ by

$$Tf := \int_a^b f(x)\, dx. \tag{2.1.4}$$

Then T is a linear map, i.e., integration is a linear operation. $\triangle$

Example 2.1.2 (Linear differential operators). For each positive integer n, let $C^n([a, b], \mathbb{R})$ be the collection of all functions $f \in \mathcal{F}([a, b], \mathbb{R})$ for which the n^{th} derivative $f^{(n)}$ exists and is continuous on $[a, b]$. (At the endpoints, these are 1-sided derivatives.) Then $C^n([a, b], \mathbb{R})$ is also a vector subspace of $\mathcal{F}([a, b], \mathbb{R})$. Define $D^n : C^n([a, b], \mathbb{R}) \to \mathcal{F}([a, b], \mathbb{R})$ by

$$D^n(f) := f^{(n)}. \tag{2.1.5}$$

Then D^n is linear, i.e., differentiation is a linear operation.

More generally, let $p_0, \dots, p_{n-1}$ be real-valued functions on $I := [a, b]$. Consider the map $T : C^n(I, \mathbb{R}) \to \mathcal{F}(I, \mathbb{R})$ defined by

$$Tf := f^{(n)} + p_{n-1}f^{(n-1)} + \cdots + p_1 f' + p_0 f. \tag{2.1.6}$$

This map T is sometimes denoted by

$$D^n + p_{n-1}D^{n-1} + \cdots + p_1 D^1 + p_0. \tag{2.1.7}$$

It is easy to verify that T is linear. We call T an n^{th}-*order linear differential operator*. Linear differential operators arise in the study of linear differential equations, which we study in Section 6.3. △

Example 2.1.3 (The Laplace transform). For fixed $c \in \mathbb{R}$, let $\mathcal{E}_c$ denote the collection of all continuous real-valued functions f on $[0, \infty)$ of *exponential order* c: there is a real constant M (depending on f) such that

$$|f(x)| \le Me^{cx} \text{ for all } x \in [0, \infty). \tag{2.1.8}$$

If M_1, M_2, and x are nonnegative, then $M_1 e^{cx} + M_2 e^{cx} \le (M_1 + M_2)e^{cx}$. It follows that $f + g$ is in $\mathcal{E}_c$ whenever both f and g are in $\mathcal{E}_c$. Similarly, $\mathcal{E}_c$ is closed under scalar multiplication. Hence, $\mathcal{E}_c$ is a vector subspace of $\mathcal{F}([0, \infty), \mathbb{R})$.

The *Laplace transform* is a map defined on $\mathcal{E}_c$. It provides a tool for solving linear initial-value problems, discussed in Chapter 6. Let $f \in \mathcal{E}_c$. Then there is a real constant M such that

$$|f(x)| \le Me^{cx} \text{ for all } x \in [0, \infty). \tag{2.1.9}$$

Hence, for any $p > 0$, we have

$$\begin{aligned} \int_0^p |e^{-st} f(t)|\, dt &\le M \int_0^p e^{-st} e^{ct}\, dt \\ &= \frac{M}{c - s}\left[e^{(c-s)p} - 1\right] \end{aligned} \tag{2.1.10}$$

Thus, for $s > c$, the limit

$$\lim_{p \to \infty} \int_0^p |e^{-st} f(t)|\, dt \tag{2.1.11}$$

exists. In particular, the improper integral

$$\int_0^\infty e^{-st} f(t)\, dt := \lim_{p \to \infty} \int_0^p e^{-st} f(t)\, dt \tag{2.1.12}$$

exists for (at least) all $s > c$. Thus, we may define $(\mathcal{L}f) : (c, \infty) \to \mathbb{R}$ by

$$(\mathcal{L}f)(s) := \int_0^\infty e^{-st} f(t)\, dt\ . \tag{2.1.13}$$

Then

$$\mathcal{L} : \mathcal{E}_c \to \mathcal{F}((c, \infty), \mathbb{R}). \tag{2.1.14}$$

The map $\mathcal{L}$ is the Laplace transform; $\mathcal{L}f$ is called the *Laplace transform of* f. Let f, g be in $\mathcal{E}_c$ and let c be a real number. Then it is clear that $\mathcal{L}(cf) = c(\mathcal{L}f)$. Also, $\mathcal{L}(f_1 + f_2)(s) = (\mathcal{L}f_1)(s) + (\mathcal{L}f_2)(s)$ whenever $s \in (c, \infty)$. Hence, $\mathcal{L}$ is a linear map from $\mathcal{E}_c$ to $\mathcal{F}((c, \infty), \mathbb{R})$. $\triangle$

Exercises for Section 2.1

Note: For *convex* and *balanced* vector spaces, see Definition B.2.3 in Appendix B.2.

2.1.1. Let X be a vector space over $\mathcal{K}$. If x, y are in X, the line segment joining x and y is the set $L[x, y] := \{ x + t(y - x) \mid 0 \le t \le 1 \}$. Show that $L[x, y]$ is convex for any x, y in X.

2.1.2. Show that an intersection of convex subsets of a vector space is also convex.

2.1.3. Let X be any vector space over $\mathcal{K}$. Assume that $B \subset X$ is balanced and $t \in \mathcal{K}$. Show that tB is also balanced.

2.1.4. Give an example of a nonbalanced subset of a vector space. Must a balanced subset of a vector space be convex?

2.1.5. Let X be a nonzero vector space over $\mathcal{K}$.

1. Find all nonempty, finite, convex subsets of X.
2. Find all nonempty, finite, balanced subsets of X.

2.1.6. Let C be a convex subset of a vector space X with scalar field $\mathcal{K}$. Show that for all $n \in \mathbb{N}$, whenever $x_1, \ldots, x_n$ are in C and $\lambda_1, \ldots, \lambda_n$ are nonnegative with $\sum_{k=1}^n \lambda_k = 1$, then $\sum_{k=1}^n \lambda_k x_k \in C$.

2.1.7. Let M, N be vector subspaces of a vector space X. Show that $M + N$ is also a vector subspace of X. When is $M \cup N$ a vector subspace of X?

2.1.8. Let X be a vector space over $\mathcal{K}$ and let $f : X \to \mathcal{K}$ be linear. Show that $f(X) = \mathcal{K}$ if and only if f is not the zero map.

2.1.9. Let X be a vector space over $\mathcal{K}$. A subset $C \subset X$ is said to be *absolutely absorbing* if for each $x \in X$, there is some $t > 0$ such that for all $\alpha \in \mathcal{K}$ with $|\alpha| \le t$, we have $\alpha x \in C$. It is said to be *absorbing* if $X = \bigcup_{n=1}^{\infty} nC$.

1. Show that if C is absolutely absorbing, then $\mathbf{0} \in C$ and C is absorbing.
2. If M is a vector subspace of X, when is M absorbing?
3. If $\{ B_\alpha \mid \alpha \in J \}$ is a collection of absorbing subsets of X, is $\bigcap_{\alpha \in J} B_\alpha$ absorbing?

2.1.10. Let $\mathcal{I}$ be the collection of all intervals of the form $[\alpha, \infty)$, where $\alpha \ge 0$ is a real number. Then for all I_1, I_2 in $\mathcal{I}$, we also have $I_1 \cap I_2 \in \mathcal{I}$. Let

$$\mathcal{F}(\mathcal{I}) := \{ f : I \to \mathbb{R} \mid I \in \mathcal{I} \} .$$

Define scalar multiplication and pointwise addition as in (2.1.2). Let $f_0 : [0, \infty) \to \mathbb{R}$ be the zero function.