1.10.8. Define $\widetilde{\mathbb{Q}}$ as in Example 1.10.6. Show that $\widetilde{\mathbb{Q}}$ with the operations of addition and multiplication defined in (1.10.23) is a field (see Section B.2).

1.10.9. 1. Let $[(c_n)] \in \widetilde{\mathbb{Q}}$ and assume that $[(c_n)] \neq [(0, 0, 0, \dots)]$. Show that there is a rational $r_0 > 0$ and a positive integer N_0 with

$$c_n < -r_0 \quad \text{for all } n \geq N_0 \quad \text{or} \quad c_n > r_0 \quad \text{for all } n \geq N_0.$$

2. For $[(a_n)]$ and $[(b_n)]$ in $\widetilde{\mathbb{Q}}$, define $[(a_n)] \leq [(b_n)]$ as in Example 1.10.6. Show that $\leq$ is a total order on $\widetilde{\mathbb{Q}}$.

1.10.10. Prove the least upper bound axiom for $\widetilde{\mathbb{Q}}$.

1.10.11. Let $\mathbb{D}$ be the collection of all $\mathbf{x} := (x_k)$ of the following form:

There is some $\tau \in \{-1, 1\}$, some nonnegative integer n, and some sequence $(a_1, a_2, a_3, \dots)$ in $\Lambda := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that

$$\begin{aligned}
x_1 &= \tau n, \\
x_2 &= \tau \left(n + \frac{a_1}{10} \right), \\
x_3 &= \tau \left(n + \frac{a_1}{10} + \frac{a_2}{10^2} \right), \\
&\vdots
\end{aligned}$$

Intuitively, each "usual" real number can be represented as some $\mathbf{x} \in \mathbb{D}$. For instance, -13.5729 corresponds to $\mathbf{x} \in \mathbb{D}$ with $\tau = -1, n = 13, a_1 = 5, a_2 = 7, a_3 = 2, a_4 = 9, a_k = 0$ for all $k \geq 5$.

Clearly, each member in $\mathbb{D}$ is a Cauchy sequence in $\mathbb{Q}$. Define $\varphi : \mathbb{D} \to \widetilde{\mathbb{Q}}$ by

$$\varphi(\mathbf{x}) := [\mathbf{x}].$$

Show that φ is onto.

1.11 Compactness

Recall from calculus that a nonempty subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$) is *compact* if it is closed and bounded. The notion of compactness is of the greatest importance. One reason is that a continuous function on a compact subset of $\mathbb{R}^n$ has a maximum value and a minimum value (Corollary 1.7.32). This statement is a powerful tool. To show that a solution to a problem exists, a standard approach is to look for a function such that the point at which the function achieves its minimum (or maximum) value would be a solution to the problem. Then, if you can prove that the domain of the function is compact, you will know that a solution exists. (See Example 1.12.12 and the proof of part 1 of Proposition 3.7.4 for an illustration of this strategy.)

Compactness is also important because every continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous.

These are such powerful results that we want to extend the notion of compactness to arbitrary metric spaces and topological spaces. Indeed, we will see in Corollary 1.11.16 that a continuous function on a compact topological space has a maximum value and a minimum value. (This result is used to prove the spectral theorem for compact operators on a Hilbert space, one of the main results of functional analysis.) Theorem 1.11.19 says that a continuous function on a compact metric space is uniformly continuous.

But for these important results to be true, we need a new definition of compactness: "closed and bounded" is not sufficient. In proving that a continuous function on a compact subset of $\mathbb{R}^n$ has maximum and minimum values, we used the statement that every sequence in a closed and bounded subset M of $\mathbb{R}^n$ has a subsequence converging to some point in M (see the proof of Corollary 1.7.32). This is not true for closed and bounded sets in an arbitrary metric space; such a set can have a sequence with no convergent subsequence.

Example 1.11.1. Consider the metric space (l^2, d) and let $\mathbf{0}$ denote the point $(0,0,0,\dots)$ in l^2. Then $\widetilde{B}(\mathbf{0},1)$, the closed ball unit in l^2 with center $\mathbf{0}$, is a closed and bounded subset of l^2. Let $\mathbf{e}_1 := (1,0,0,\dots), \mathbf{e}_2 := (0,1,0,0,\dots), \dots$. Then $(\mathbf{e}_n)$ is a sequence in $\widetilde{B}(\mathbf{0},1)$ but $d(\mathbf{e}_n, \mathbf{e}_m) = \sqrt{2}$ for all $n \neq m$. Thus, $(\mathbf{e}_1, \mathbf{e}_2, \dots)$ has no convergent subsequence. $\triangle$

In this section we discuss three definitions of a compact metric space:

1. A metric space X is compact if every sequence in X has a subsequence converging to a point in X.
2. A metric space is compact if it is totally bounded and complete.
3. A metric space X is compact if every open cover for X contains a finite subcover.

We will restate these definitions and prove that they are equivalent after introducing and discussing the concepts "totally bounded" and "open cover".

We will adopt statement 3 above as our definition of compactness for arbitrary topological spaces. Note that any concept (in this section and elsewhere) initially defined for a topological space X automatically applies to subsets of X, considered as topological subspaces of X

Totally bounded metric spaces

A metric space X is totally bounded if and only if for every $r > 0$ it can be covered by *finitely* many open balls in X, all of radius r.

Definition 1.11.2 (Totally bounded metric space). A metric space X is *totally bounded* if for every $r > 0$ there is a finite $S \subset X$ such that

$$X = \bigcup_{z \in S} B(z, r). \tag{1.11.1}$$