Chapter 1

Metric and topological spaces

In this chapter we will study metric spaces – the most general spaces in which we can measure distances between points. In Chapter 2 we will see how to make this structure interact with vector space structure to produce normed spaces, the main actors in functional linear analysis.

We also discuss open and closed sets, topological spaces, and the concepts of continuity, convergence of sequences, density, separability, and compactness. The last two sections introduce two important results of functional analysis: the fixed point theorem and Baire's category theorem.

1.1 Metrics and metric spaces

The concept of "distance between two points" is needed virtually everywhere in analysis. Consider the concept of continuity as studied in calculus. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \epsilon. \tag{1.1.1}$$

This says that the *distance* between $f(x)$ and $f(a)$ can be made arbitrarily small by requiring x to be sufficiently close to a. Similarly, a sequence $(a_1, a_2, \dots)$ of real numbers converges to l if given $\epsilon > 0$, there is some positive integer N such that

$$n \geq N \implies |a_n - l| < \epsilon; \tag{1.1.2}$$

eventually, the distance between the terms of the sequence and l can be made arbitrarily small.

If x, y are points in an arbitrary set, how should the distance between them be defined? What properties are needed to make such a concept both natural and useful? Consider the set of real numbers. The distance between real numbers a

and b is defined to be $|a-b|$. First, $|a-b| \geq 0$. Second, $|a-b| = |b-a|$. Third, $|a-b| = 0$ if and only if $a = b$. Fourth, for any three numbers a, b, c, we have $|a-c| \leq |a-b| + |b-c|$. This last property, while less obvious than the others, is very useful in proving many important results in calculus.

These properties are all natural because of our concept of "distance". If we now denote the number $|a-b|$ by $d(a,b)$, these properties can be written in terms of a distance function d, called a "metric".

For any set X, let $X^2 := X \times X$ be the set $\{ (x,y) \mid x, y \in X \}$.

Definition 1.1.1 (Metric). Let X be any nonempty set. A *metric* or *distance function* on X is a function $d : X^2 \to [0, \infty)$ satisfying the following properties:

1. $d(x,y) = 0$ if and only if $x = y$.
2. *Symmetry*: $d(x,y) = d(y,x)$ for all $x, y \in X$.
3. *Triangle inequality*: $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Note that since d is a function from X^2 to $[0, \infty)$, we always have $d(a,b) \geq 0$.

Property 3, the triangle inequality, recognizes the expression "distance as the crow flies": the shortest distance between two points is the straight line connecting them.

Proposition 1.1.2. *Let d be a metric on a nonempty set X. For $x, y, z \in X$,*

$$|d(x,z) - d(y,z)| \leq d(x,y). \tag{1.1.3}$$

PROOF. By the triangle inequality,

$$d(x,z) - d(y,z) \leq d(x,y) \quad \text{and} \quad d(y,z) - d(x,z) \leq d(y,x) \tag{1.1.4}$$

for all x, y and z in X. Hence, $|d(x,z) - d(y,z)| \leq d(x,y)$. □

Often a number of interesting metrics can be defined on a set X; a metric emphasizes some feature of interest while ignoring others. For instance, let X be the United States. Three possible metrics are d_g, which measures geographical distance; d_c, which measures travel cost; and d_t, which measures travel time. (Using d_t, New York City is closer to Los Angeles than it is to Champaign, IL, unless one can travel by private jet.) Indeed, travel web sites like Orbitz and Travelocity let the user sort flights by distance in time and distance in money.

More generally, we may think of "distance" as quantifying a *difference*. What is the distance between two species of animals? The metric d could measure how far back one must go to find a common ancestor. Or it could measure the percentage of DNA the two species have in common. Creationists might say that the distance is the same between all pairs of species, since they believe that all species were created at the same time and are therefore unrelated; this way of measuring distance is the *discrete* metric.

Definition 1.1.3 (Discrete metric). The *discrete metric* on a nonempty set X is the function $d: X^2 \to [0, \infty)$ defined by

$$d(x, y) := \begin{cases} 0 & , \text{ if } x = y \\ 1 & , \text{ if } x \neq y. \end{cases} \tag{1.1.5}$$

(As noted in Chapter 0, we use the symbol := to denote "equal by definition".) The discrete metric is an unnatural way to measure geographical distance or the distance between real numbers. But it is a natural – if uninformative – way to measure differences: it says that either two points are the same or they are different.

Remark 1.1.4. Infinitely many metrics can be defined on any nonempty set: let d be the discrete metric and for $k > 0$, define $D_k : X^2 \to [0, \infty)$ by $D_k(x, y) := kd(x, y)$. Then D_k is also a metric on X. △

Definition 1.1.5 (Metric space). A *metric space* is a pair (X, d), where X is a nonempty set and d is a metric on X.

When it is clear what the metric is, we may write X rather than (X, d) and speak of "the distance between a and b in X". In such a case we always mean the distance *with respect to* the metric defined on X.

Some important metric spaces

It will be easy to see that the metrics we consider in this chapter satisfy the first two properties of Definition 1.1.1, but verifying the triangle inequality can be tricky; the inequalities we proved in Chapter 0 are indispensable.

Example 1.1.6 (Two different metrics for $\mathbb{R}^n$). As discussed in Appendix B.5, throughout this book, we will denote an element of $\mathbb{R}^n$ either by a bold letter or by a column matrix whose n entries are indexed non-bold versions of the same letter.

Here are two different metrics we can assign to $\mathbb{R}^n$:

1. The function $d: \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by

$$d(\mathbf{x}, \mathbf{y}) := \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}} \tag{1.1.6}$$

is the *standard metric* on $\mathbb{R}^n$, also called the *Euclidean metric*. Showing that d satisfies the first two properties of Definition 1.1.1 is straightforward, so we will only show that the triangle inequality is satisfied.

Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be in $\mathbb{R}^n$. Then $(x_i - z_i)^2 = |x_i - z_i|^2 \leq (|x_i - y_i| + |y_i - z_i|)^2$ for all $1 \leq i \leq n$. Hence, by Minkowski's inequality (Inequality 0.5),

$$\begin{aligned} d(\mathbf{x}, \mathbf{z}) := \left(\sum_{i=1}^{n} (x_i - z_i)^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|)^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} |y_i - z_i|^2\right)^{\frac{1}{2}} := d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}). \end{aligned} \tag{1.1.7}$$

2. Another metric on $\mathbb{R}^n$ is the function $D : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by

$$D(\mathbf{x}, \mathbf{y}) := \sum_{k=1}^{n} |x_k - y_k|. \tag{1.1.8}$$

Again, it is clear that the first two properties in Definition 1.1.1 are satisfied. For the triangle inequality, note that if $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are in $\mathbb{R}^n$, then

$$\begin{aligned} D(\mathbf{x}, \mathbf{z}) := \sum_{k=1}^{n} |x_k - z_k| &\leq \sum_{k=1}^{n} (|x_k - y_k| + |y_k - z_k|) \\ &= \sum_{k=1}^{n} |x_k - y_k| + \sum_{k=1}^{n} |y_k - z_k| \\ &:= D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}). \end{aligned} \tag{1.1.9}$$

For $n = 3$, what is the distance between $(0, 0, 0)$ and $(1, 1, 1)$? Using the standard metric, the distance is $\sqrt{3}$. Using the metric D above, it is 3.

We could assign other metrics to $\mathbb{R}^n$. But unless stated otherwise, when we consider $\mathbb{R}^n$ as a metric space, we mean $\mathbb{R}^n$ with the standard Euclidean metric. Consequently, often we will simply write "the metric space $\mathbb{R}^n$". △

In the next examples we define the sets $\mathbb{C}^n$, l^p, l^∞, and $C[a, b]$ and assign a metric to each, turning it into a metric space. Like $\mathbb{R}^n$, these spaces appear frequently in this book. When we speak of $\mathbb{C}^n$, l^p, l^∞, or $C[a, b]$ as a metric space, we mean the set together with the metric defined below.

Example 1.1.7 ($\mathbb{C}^n$ with the standard metric). We denote an element of the set $\mathbb{C}^n$ by a bold letter or a column matrix: if $\mathbf{z} \in \mathbb{C}^n$, then $\mathbf{z} := \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$. The standard (Euclidean) metric on $\mathbb{C}^n$ is the function $d : \mathbb{C}^n \times \mathbb{C}^n \to [0, \infty)$ defined by

$$d(\mathbf{z}, \mathbf{w}) := \left(\sum_{j=1}^{n} |z_j - w_j|^2\right)^{\frac{1}{2}}. \qquad \triangle \tag{1.1.10}$$