Chapter 0

Three important inequalities

In this short chapter we prove three inequalities that will be needed for some basic results in functional analysis. This chapter is more technical than the others, and you may wish to skip ahead to Chapter 1 on metric and topological spaces to get a better feeling for the tone of the book. But you will soon need to return to this chapter, since Minkowski's inequality (Inequality 0.5) is needed to show that various functions are metrics. The other two of our "three important inequalities" are used to prove Minkowski's inequality. They also show up in their own right in later chapters.

Lemma 0.1. *Let* $0 < \lambda < 1$*. Then*

$$t^\lambda \leq 1 - \lambda + \lambda t \quad \textit{for all } t \geq 0. \tag{0.1}$$

This inequality becomes an equality only when $t = 1$*.*

Remark. Below and elsewhere we use := to denote "equal by definition"; we use = to denote "can be shown to be equal". △

Proof. Define $f : [0, \infty) \to \mathbb{R}$ by

$$f(t) := 1 - \lambda + \lambda t - t^\lambda, \tag{0.2}$$

whose graph is shown in Figure 1. Then

$$f'(t) = \lambda - \lambda t^{\lambda - 1} = \lambda \left(1 - \frac{1}{t^{1-\lambda}}\right). \tag{0.3}$$

Thus,

$$f'(t) \begin{cases} > 0\,, & \text{if } t > 1 \\ < 0\,, & \text{if } 0 < t < 1. \end{cases} \tag{0.4}$$

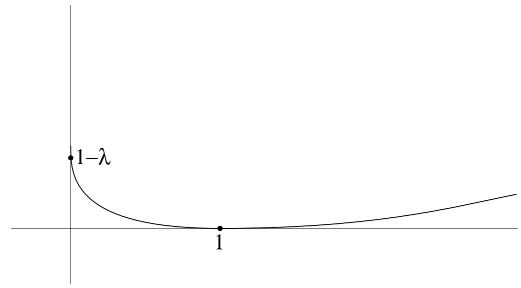

Figure 1: Graph of $f(t) := 1 - \lambda + \lambda t - t^\lambda$, with minimum value 0 at $t = 1$.

Consequently, $0 = f(1)$ is the minimum value of f. Therefore, $f(t) \geq 0$ for all $t \geq 0$, with equality if and only if $t = 1$. Hence,

$$\begin{aligned} t^\lambda &\leq 1 - \lambda + \lambda t \quad \text{for all } t \geq 0 \\ t^\lambda &= 1 - \lambda + \lambda t \quad \text{if and only if } t = 1. \qquad \square \end{aligned} \tag{0.5}$$

The three inequalities that follow all use the notion of *conjugate exponents*.

Definition 0.2 (Conjugate exponent). Positive real numbers p, q such that

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{0.6}$$

are called *conjugate exponents*. The pair $1, \infty$ is also considered to be a pair of conjugate exponents, since $p \to 1$ implies $q \to \infty$.

If p, q are integers, the only pair of conjugate exponents is $2, 2$.

Inequality 0.3. Let p, q be conjugate exponents with $1 < q < \infty$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for all} \quad a, b \geq 0. \tag{0.7}$$

Equality holds if and only if $a^p = b^q$.

Proof. Assume p, q are as above and $a, b \geq 0$. If $b = 0$, there is nothing to prove. Assume $b > 0$. In (0.1), substitute $\frac{1}{p} < 1$ for λ and $a^p b^{-q}$ for t to obtain

$$ab^{-\frac{q}{p}} = (a^p b^{-q})^{\frac{1}{p}} \leq 1 - \frac{1}{p} + \frac{1}{p} a^p b^{-q}. \tag{0.8}$$

So

$$ab^{-\frac{q}{p}} \leq \frac{1}{p} a^p b^{-q} + \frac{1}{q}. \tag{0.9}$$

Therefore,

$$ab^{-\frac{q}{p}+q} \;\leq\; \frac{a^p}{p} + \frac{b^q}{q}. \tag{0.10}$$

But $-\frac{q}{p} + q = 1$, so

$$ab \;\leq\; \frac{a^p}{p} + \frac{b^q}{q}, \tag{0.11}$$

with equality if and only if $1 = t = a^p b^{-q}$, i.e., $a^p = b^q$. $\square$

Inequality 0.4 (Hölder's inequality). Let p, q be conjugate exponents with $1 < q < \infty$. For any integer $n \geq 1$, assume that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are nonnegative. Then

$$\sum_{k=1}^{n} a_k b_k \;\leq\; \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}} \tag{0.12}$$

Proof. Let p, q, n, $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be as above. Let

$$A := \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}}, \quad B := \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}. \tag{0.13}$$

If $AB = 0$, clearly (0.12) is satisfied, so assume $AB > 0$. Observe that

$$\sum_{k=1}^{n} \frac{a_k^p}{A^p} = 1 = \sum_{k=1}^{n} \frac{b_k^q}{B^q}. \tag{0.14}$$

Next, apply Inequality 0.3 to get

$$\frac{a_k}{A}\frac{b_k}{B} \;\leq\; \frac{a_k^p}{pA^p} + \frac{b_k^q}{qB^q} \qquad \text{for all } k = 1, \ldots, n. \tag{0.15}$$

Thus we have

$$\sum_{k=1}^{n} \frac{a_k b_k}{AB} \;\leq\; \frac{1}{p} + \frac{1}{q} = 1, \tag{0.16}$$

and hence,

$$\sum_{k=1}^{n} a_k b_k \;\leq\; AB = \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}. \qquad \square \tag{0.17}$$