Since $f_n(X) \subset S$ and S is closed in (Y, ρ), equation (D.6.1) implies that

$$g(x) = \lim_{n\to\infty} f_n(x) \in \overline{S} = S \quad \text{for all } x \in X.$$

Hence, $g(X) \subset S$ so that $g \in M$. So, M is closed in $C(X,Y)$. Because $C(X,Y)$ is complete, M must also be complete (by Proposition 1.8.8).

6.3.9 Let $y_1(x) := e^{\alpha x} \cos \beta x$. Then

$$\begin{aligned} y_1'(x) &= e^{\alpha x}(\alpha \cos \beta x - \beta \sin \beta x), \\ y_1''(x) &= e^{\alpha x}((\alpha^2 - \beta^2) \cos \beta x - 2\alpha\beta \sin \beta x). \end{aligned} \tag{D.6.2}$$

Hence,

$$y_1''(x) + by_1'(x) + cy_1(x) = e^{\alpha x}((\alpha^2 - \beta^2 + b\alpha + c) \cos \beta x - (2\alpha\beta + b\beta) \sin \beta x).$$

Since $b^2 - 4c < 0$, we have $4c - b^2 > 0$ and the quadratic formula yields

$$\alpha = \frac{-b}{2} \quad \text{and} \quad \beta = \frac{\sqrt{4c - b^2}}{2}.$$

Hence,

$$\begin{aligned} \alpha^2 - \beta^2 + b\alpha + c &= \frac{b^2}{4} + \frac{b^2 - 4c}{4} - \frac{b^2}{2} + c = 0 \\ 2\alpha\beta + b\beta &= \frac{-b\sqrt{4c - b^2}}{2} + \frac{b\sqrt{4c - b^2}}{2} = 0. \end{aligned}$$

Thus, (D.6.2) implies that y_1 is a solution to $y'' + by' + cy = 0$. Similarly, let $y_2(x) := e^{\alpha x} \sin \beta x$. Then

$$\begin{aligned} y_2'(x) &= e^{\alpha x}(\alpha \sin \beta x + \beta \cos \beta x) \\ y_2''(x) &= e^{\alpha x}((\alpha^2 - \beta^2) \sin \beta x + 2\alpha\beta \cos \beta x), \end{aligned}$$

so

$$y_2''(x) + by_2'(x) + cy_2(x) = e^{\alpha x}((\alpha^2 - \beta^2 + b\alpha + c) \sin \beta x + (2\alpha\beta + b\beta) \cos \beta x) = 0.$$

Thus, y_2 is also a solution to $y'' + by' + cy = 0$. Finally, suppose that $c_1 y_1 + c_2 y_2 = \mathbf{0}$ for some real numbers c_1, c_2. Then $c_1 \cos \beta x + c_2 \sin \beta x = 0$ for all $x \in \mathbb{R}$. Substituting $x := 0$ gives $c_1 = 0$. Thus, $c_2 = 0$ also. Hence, $\{y_1, y_2\}$ is linearly independent. Thus, by Theorem 6.3.28, $\{y_1, y_2\}$ forms a basis for the solutions space of $y'' + by' + cy = 0$.

6.3.11 1. The proof will be by induction on n. For $n = 1$, this is trivial. Assume that whenever $r_1, \ldots, r_n$ are distinct real numbers, then $\{E_{r_1}, \ldots, E_{r_n}\}$ is linearly independent. Let $b_1, \ldots, b_{n+1}$ be distinct real numbers. Denote the zero function on $\mathbb{R}$ and the constant function $x \mapsto 1$ on $\mathbb{R}$ by $\mathbf{0}$ and $\mathbf{1}$, respectively. Suppose that $c_1, \ldots, c_n, c_{n+1}$ are real numbers such that

$$c_1 E_{b_1} + \cdots + c_n E_{b_n} + c_{n+1} E_{b_{n+1}} = \mathbf{0}. \tag{D.6.3}$$

Put $r_k := b_k - b_{n+1}$ for $1 \leq k \leq n$. When we divide by $E_{b_{n+1}}$, equation (D.6.3) becomes

$$c_1 E_{r_1} + \cdots + c_n E_{r_n} = -c_{n+1} \mathbf{1}. \tag{D.6.4}$$

Thus, taking derivative in both sides of (D.6.4), we obtain

$$c_1 r_1 E_{r_1} + \cdots + c_n r_n E_{r_n} = \mathbf{0}.$$

Now, $r_1, \ldots, r_n$ are all distinct and nonzero (because $b_1, \ldots, b_{n+1}$ are all distinct), the inductive hypothesis implies that $c_1 r_1 = \cdots = c_n r_n = 0$. Hence, $c_1 = \cdots = c_n = 0$. It now follows from (D.6.3) that $c_{n+1} = 0$ also. Hence, $\{E_{b_1}, \ldots, E_{b_n}, E_{b_{n+1}}\}$ is linearly independent.

2. We will show that each E_{r_k} is a solution to (6.3.81) on $(-\infty, \infty)$. For each $1 \leq \ell, k \leq n$, we have

$$E_{r_k}^{(\ell)} = r_k^\ell E_{r_k}.$$

Hence, each E_{r_k} is a solution to (6.3.81), since

$$\begin{aligned} & E_{r_k}^{(n)} + a_{n-1} E_{r_k}^{(n-1)} + \cdots + a_1 E_{r_1}' + a_0 E_{r_k} \\ & = r_k^n E_{r_k} + a_{n-1} r_k^{n-1} E_{r_k} + \cdots + a_1 r_k E_{r_k} + a_0 E_{r_k} \\ & = (r_k^n + a_{n-1} r_k^{n-1} + \cdots + a_1 r_k + a_0) E_{r_k} = \mathbf{0}. \end{aligned}$$

Thus, by part 1, we now know that $\{E_{r_1}, \ldots, E_{r_n}\}$ is a linearly independent set of n solutions to (6.3.81). Hence, by Theorem 6.3.28, $\{E_{r_1}, \ldots, E_{r_n}\}$ is a basis for the solutions space of (6.3.81). The desired conclusion follows.

6.3.13 For each integer $1 \leq k \leq n$, let $f_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f_k \left(t, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) := e^{t^2} \frac{x_k}{1 + x_1^2 + \cdots + x_n^2}.$$

Then for any $t \in \mathbb{R}$, the function $\frac{\partial f_k}{\partial x_j}(t, \cdot)$ is continuous on $\mathbb{R}^n$ for all $1 \leq k, j \leq n$. So, by Lemma 6.3.10, each f_k is Lipschitz with respect to the variable in $\mathbb{R}^n$. Put $\mathbf{f} := \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$. Then by Remark 6.3.9, $\mathbf{f}$ is Lipschitz with respect to to the variable in $\mathbb{R}^n$. Now, apply Theorem 6.3.12 with $t_0 := 0$ and $\mathbf{z}_0 := \begin{pmatrix} 1 \\ \vdots \\ n \end{pmatrix}$.

6.3.15 No. Let $a_{m-1}, \ldots, a_1, a_0$ be as in the statement of the problem. For any continuous $h : (-1, 1) \to (-1, 1)$, consider the differential equation

$$y^{(m)} + a_{m-1} y^{(m-1)} + \cdots + a_1 y' + a_0 y = h. \tag{D.6.5}$$

Suppose on the contrary that there are f_1 and f_2 satisfying (D.6.5). Then $f_1 - f_2$ is a solution to the homogeneous differential equation

$$y^{(m)} + a_{m-1} y^{(m-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{D.6.6}$$

By assumption, the equation $r^m + a_{m-1} r^{m-1} + \cdots + a_1 r + a_0 = 0$ has m distinct real roots. Hence, by Exercise 6.3.11, the solutions set to (D.6.6) has a basis consisting of the functions $E_{r_j} : t \mapsto e^{r_j t}, 1 \leq j \leq m$. Thus, $f_1 - f_2$ can be written as a linear combination of $E_{r_j}, 1 \leq j \leq m$. This is a contradiction, since

$$f_1(t) - f_2(t) = \cos^2 t - \sin^2 t = \cos(2t) \quad \text{for all } t \in (-1, 1).$$