Appendix D

Solutions to odd exercises

D.1 Solutions for Chapter 1

1.1.1 Yes. Let $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ be in $\mathbb{R}^n$. Since $|x_k - z_k| \le |x_k - y_k| + |y_k - z_k|$, we must have

$$\max_{1\le k\le n} |x_k - z_k| \le \max_{1\le k\le n} (|x_k - y_k| + |y_k - z_k|) \le \max_{1\le k\le n} |x_k - y_k| + \max_{1\le k\le n} |y_k - z_k|.$$

Thus, $d_\infty(\mathbf{x}, \mathbf{z}) \le d_\infty(\mathbf{x}, \mathbf{y}) + d_\infty(\mathbf{y}, \mathbf{z})$.

1.1.3 No. Consider the Euclidean metric d on $\mathbb{R}$ defined by $d(a, b) := |b - a|$. Then $D(0, 2) = 4 > 1 + 1 = D(0, 1) + D(1, 2)$.

1.1.5 The function d_f is a metric on X if and only if f is 1–1. Assume first that f is 1–1. Let x, y, z be in X. Then $d_f(x, y) \ge 0$ and

$$d_f(x, y) = 0 \iff d(f(x), f(y)) = 0 \iff f(x) = f(y) \iff x = y.$$

Also, since d is a metric, we have $d_f(x, y) = d(f(x), f(y)) = d(f(y), f(x)) = d_f(y, x)$. Finally, the triangle inequality for d implies that

$$d_f(x, z) = d(f(x), f(z)) \le d(f(x), f(y)) + d(f(y), f(z)) = d_f(x, y) + d_f(y, z).$$

On the other hand, assume that f is not 1–1. Then there are distinct a, b in X with $f(a) = f(b)$. Hence, $d_f(a, b) = d(f(a), f(b)) = 0$ so that d_f is not a metric on X.

1.1.7 Yes. To show that ρ satisfies the triangle inequality, consider the function $f(t) := \frac{t}{a+t}$ for all $t \in [0, \infty)$. Then $f'(t) = \frac{a}{(a+t)^2} > 0$ for all $t \in [0, \infty)$. So f is increasing on $[0, \infty)$. Thus, if u, s, and t are in $[0, \infty)$ and $u \le s + t$, then

$$\frac{u}{a+u} \le \frac{s+t}{a+s+t} = \frac{s}{a+s+t} + \frac{t}{a+s+t} \le \frac{s}{a+s} + \frac{t}{a+t}.$$

Hence, for all x, y and z in X, we must have

$$\frac{d(x, z)}{a + d(x, z)} \le \frac{d(x, y)}{a + d(x, y)} + \frac{d(y, z)}{a + d(y, z)}.$$

1.1.9 1. If (x_k) is a sequence of complex numbers converging to $l \in \mathbb{C}$, then there is some positive integer N such that $|x_k - l| < 1$ for al $k > N$. Put $M := \max_{1\le k\le N} |x_k|$. Then $|x_k| \le \max\{M, |l|+1\}$ for all k. Hence, $(x_k) \in l^\infty$. The constant sequence $(1,1,1,\dots)$ is in c but not in l^p for any $1 \le p < \infty$.

2. From part 1, $c_0 \subset l^\infty$. However, c_0 is not bounded in l^∞ because $\mathbf{x}_n := (n,0,0,\dots)$ is in c_0 for all $n \in \mathbb{Z}$, but the distance between $\mathbf{x}_n$ and $\mathbf{x}_0$ in l^∞ is n.

1.1.11 Yes. Let $1 \le p$ and let $\mathbf{x} := (x_k) \in l^p$. Then $\sum_{k=1}^\infty |x_k|^p < \infty$. Hence, necessarily, we must have $|x_k| \to 0$ as $k \to \infty$. Thus, $\sup_{k\ge1} |x_k| < \infty$ and so, $\mathbf{x} \in l^\infty$. Hence, $l^p \subset l^\infty$. If $p \le q < \infty$, then since $|x_k| \to 0$ as $k \to \infty$, we have $|x_k|^p \le |x_k|^q$ for all k sufficiently large. Hence, $\sum_{k=1}^\infty |x_k|^q < \infty$. Thus, $\mathbf{x} \in l^q$. Hence, $l^p \subset l^q$.

1.1.13 Let x, y be in $B(x_0, r)$. Then $d(x,y) \le d(x,x_0)+d(x_0,y) < r+r = 2r$. Hence, $\operatorname{diam} B(x_0,r) := \sup\{\, d(x,y) \mid x, y \text{ in } B(x_0,r)\,\} \le 2r$. If d is the discrete metric on X and $0 < r \le 1$, then $B(x_0,r) = \{x_0\}$, and thus, $\operatorname{diam} B(x_0,r) = 0$.

1.1.15 Yes. To see why, define $f : l^p \to l^p$ by $f(\mathbf{x}) := \left(\frac{x_k}{\alpha}\right)$ for all $\mathbf{x} := (x_k) \in l^p$. Then f is a bijection. Also, $D(f(\mathbf{x}), f(\mathbf{y})) = d(\mathbf{x},\mathbf{y})$ because

$$D(f(\mathbf{x}), f(\mathbf{y})) = D\left(\left(\frac{x_k}{\alpha}\right), \left(\frac{y_k}{\alpha}\right)\right) = \alpha\left(\sum_{k=1}^{\infty}\left|\frac{x_k}{\alpha} - \frac{y_k}{\alpha}\right|^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{\infty}|x_k - y_k|^p\right)^{\frac{1}{p}}.$$

1.2.1 No. Let X be any infinite set and let d be the discrete metric on X. Then $B(x, \frac{1}{2}) = \{x\}$ for any $x \in X$.

1.2.3 Let d be a discrete metric on a nonempty set X. Let $S \subset X$. If $S = \emptyset$, then S is open in (X,d). Assume that $S \ne \emptyset$. Let $x \in S$. Then since $d(y,x) = 1$ for all $y \in X$ with $y \ne x$, we have $B(x, \frac{1}{2}) = \{x\} \subset S$. Hence, every point in S is an interior point of S. Thus, S is open in (X,d).

1.2.5 Let (X,d) be a metric space. Then $B(x,r) = X$ for all $x \in X$ and all $r > 0$ if and only if X contains exactly one point.

1.2.7 Let $\mathbf{z} \in M$. Then there is some n such that $\mathbf{z} = (z_1, \dots, z_n, 0, 0 \dots)$. Let $r > 0$. Choose a sequence (ϵ_k) of positive numbers such that $\sum_{k=1}^\infty \epsilon_k^p < r^p$. Set

$$\mathbf{a} := (z_1, \dots, z_n, \epsilon_1, \epsilon_2, \epsilon_3 \dots).$$

Then $\mathbf{a} \in l^p$ and

$$d_p(\mathbf{z},\mathbf{a}) = \begin{cases} \left(\sum_{k=1}^n \epsilon_k^p\right)^{\frac{1}{p}} & \text{, if } p < \infty \\ \sup_{k\ge1} \epsilon_k & \text{, if } p = \infty \end{cases}$$

In either case, $d_p(\mathbf{z},\mathbf{a}) < r$. So, $B(\mathbf{z}, r)$ contains the point $\mathbf{a}$ and $\mathbf{a} \notin M$. Thus, since $r > 0$ was arbitrary, $\mathbf{z}$ is not an interior point of M, so M has no interior points.

1.2.9 Let $\mathbf{a} := (a_1, a_2, a_3, \dots) \in l^p$ and let $r > 0$. For each $t \in (0,r)$, put

$$\mathbf{u}_t := (a_1 + t, a_2, a_3, \dots).$$

Then $\mathbf{u}_t \in l^p$ and $d_p(\mathbf{a}, \mathbf{u}_t) = t < r$ for all $t \in (0,r)$. Hence, $\mathbf{u}_t \in B(\mathbf{a}, r)$ for all $t \in (0,r)$.