Appendix C

Some technical results

C.1 Lemma used to prove Theorem 4.1.14

Lemma C.1.1. *Let* $\| \ \|$ *be a norm on a vector space* X *over* $\mathcal{K}$ *such that*

$$\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2) \quad \textit{for all } a, b \textit{ in } X \tag{C.1.1}$$

Define $F : X^2 \to \mathbb{R}$ *by*

$$F(x,y) := \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big) \quad \textit{for all } (x,y) \in X^2. \tag{C.1.2}$$

Then for all x, y, z *in* X *and* $\lambda \in \mathcal{K}$*, we have*

1. $F(x+z, y) = F(x,y) + F(z,y)$
2. $8\,(F(\lambda x, y) + F(x, \lambda y)) = (|\lambda+1|^2 - |\lambda-1|^2)(\|x+y\|^2 - \|x-y\|^2)$.

Proof. 1. Using (C.1.1), we have $2F(x,y) = \|x+y\|^2 - \|x\|^2 - \|y\|^2$. Thus, $F(x+z,y) = F(x,y) + F(z,y)$ if and only if

$$\|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x+z\|^2 + \|z+y\|^2 + \|x+y\|^2. \tag{C.1.3}$$

We will prove that (C.1.3) holds. Denote by S the right side of (C.1.3). Then

$$\begin{aligned}
2S &= 2\|x+z\|^2 + 2\|z+y\|^2 + 2\|x+y\|^2 \\
&= \|(x+z)+(z+y)\|^2 + \|(x+z)-(z+y)\|^2 + 2\|x+y\|^2 \\
&= \|2z+x+y\|^2 + \|x-y\|^2 + 2\|x+y\|^2 \\
&= \|2z+x+y\|^2 + (2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2) + 2\|x+y\|^2 \qquad \text{(C.1.4)} \\
&= \|2z+x+y\|^2 + \|x+y\|^2 + 2\|x\|^2 + 2\|y\|^2 \\
&= \frac{1}{2}\left(\|(2z+x+y)+(x+y)\|^2 + \|(2z+x+y)-(x+y)\|^2\right) + 2\|x\|^2 + 2\|y\|^2 \\
&= 2\|x+y+z\|^2 + 2\|z\|^2 + 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$