Vector Calculus, Linear Algebra and Differential Forms: A Unified Approach
5th Edition, First Printing

Complete List of Errata and Notes as of September 17, 2020

You have a copy of the first printing if the copyright page contains, after "Printed in the United States of America" the numbers

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You have a copy of the second printing if the numbers end with 2:

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New items are marked in red (some marked in red were previously posted in the list for the second printing). The list is divided into three sections: errata; typos and minor minor; and notes and clarifications.

Errata

Page 30 [added April 9, 2019] In the caption for Figure 0.76, $(z) = z^2$ should be $f(z) = z^2$.

Page 73 [added Sept. 15, 2017] Proposition 1.4.11: "$\vec{\mathbf{a}}$ a vector in $\mathbb{R}^m$" should be "$\vec{\mathbf{b}}$ a vector in $\mathbb{R}^m$".

Page 79 The far right side of equation 1.4.52 should be $|\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})|$, not $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})|$; the volume is a nonnegative number.

Page 97 [added April 9, 2019] Part 5 of Theorem 1.5.29: "If h is continuous at $\mathbf{x}_0 \in \overline{U}$" should be "If h is defined and continuous at $\mathbf{x}_0 \in \overline{U}$".

Page 107 [added Sept. 29, 2017] 5th paragraph, line 7: "saying that the x_m have a limit in $[0, 1/2)$" should be " … have a limit in $[0, 1)$".

Page 110 Inequalities 1.6.14: The second $|\mathbf{x}_{i_j} - \mathbf{a}| < \delta$ should be $|\mathbf{y}_{i_j} - \mathbf{a}| < \delta$.

PAGE 113 Fifth paragraph, first line: "The disc $\{ z \in \mathbb{C} \mid |z| \le R \}$", not "The disc $\{ z \in \mathbb{C} \mid z \le R \}$"

PAGE 132 In the margin note that starts "We write equation 1.7.49" we should have referred to "lengths of matrices" rather than "absolute values".

PAGE 134 In the last equation of the proof (just before the exercises) $[0]$ should be 0 (the 0 is scalar, not a matrix).

PAGE 145 [added July 9, 2019] Theorem 1.9.1: "there exists $\mathbf{c}_o \in (\mathbf{a}, \mathbf{b})$" (open interval) not "there exists $\mathbf{c}_o \in [\mathbf{a}, \mathbf{b}]$" (closed interval).

PAGE 150 The proof of Theorem 1.9.8. is an application of Theorem 1.6.13, the mean value theorem in one variable, not an application of Theorem 1.9.1.

PAGE 170 Corollary 2.2.7 also uses Proposition 1.3.13; the proof of that proposition shows that if the linear transformation "multiplication by A" is bijective, its inverse mapping is linear and so has a matrix; this matrix is a two-sided inverse of A.

PAGE 170 In the margin we define a *pivotal row* by saying say that when a row of $\widetilde{A}$ contains a pivotal 1, the corresponding row of A is a pivotal row. Because we use row operations when row reducing a matrix, "corresponding row" is not well defined. We will avoid the term in the future (although one could define pivotal row using the transpose).

PAGE 171 To avoid the use of "pivotal row" (see the erratum for page 170), we will replace formula 2.2.11 by

$$\begin{aligned} A \text{ one to one} &\iff \text{every column of } \widetilde{A} \text{ contains a pivotal 1} \\ &\iff \text{every row of } \widetilde{A} \text{ contains a pivotal 1} \\ &\iff A \text{ is onto.} \end{aligned} \qquad 2.2.11$$

and we will change the first margin note to " ... which has a pivotal 1 in each column but not in each row".

PAGE 187 There are several mistakes of sign in the third displayed equation of Example 2.4.17. It should be

$$\begin{aligned} \int_0^\pi \sin nx \sin mx \, dx &= \frac{1}{2} \int_0^\pi \Big((\cos(n-m)x - \cos(n+m)x) \Big) \, dx \\ &= \frac{1}{2} \left(\left[\frac{\sin(n-m)x}{n-m} \right]_0^\pi - \left[\frac{\sin(n+m)x}{n+m} \right]_0^\pi \right) = 0. \end{aligned}$$

PAGE 187 Next-to-last line, in the displayed equation,

$$+ \dots a_k \sin n_k x \quad \text{should be } + \dots + a_k \sin n_k x.$$

Page 188 [new, posted Sept. 17, 2020] We will change the subheading at the bottom of the page to "Proof of Proposition and Definition 2.4.11".

Page 194 Two lines before equations 2.5.6: "for $\vec{\mathbf{v}}_1$ ... the first entry is -1" should be "for $\vec{\mathbf{v}}_1$ ... the first entry is -2". " ... the corresponding entries for $\vec{\mathbf{v}}_2$ are -3, -2, and 0" should be " ... the corresponding entries for $\vec{\mathbf{v}}_2$ are are -1, 1, and 0."

Page 197 Line -4: To avoid the use of "pivotal row" (see the erratum for page 170), we will replace "the pivotal rows of A are linearly independent" by "the rows of $\widetilde{A}$ containing pivotal 1's are linearly independent".

Page 212 In the second printing of the text, we will move the discussion of dimension, including Proposition and Definition 2.6.21, immediately before the subsection "Matrix with respect to a basis and change of basis". In Proposition and Definition 2.6.17 on the change of basis matrix, V is said to be n-dimensional, but with the current order of discussion, dimension isn't yet defined, and we don't yet know that $\{\mathbf{v}\}$ and $\{\mathbf{v}'\}$ both have n elements. Fortunately, the proof of Proposition and Definition 2.6.21 requires only Proposition 2.6.15 and the fact that $\Phi^{-1}_{\{\mathbf{w}\}}$ and $\Phi^{-1}_{\{\mathbf{v}\}}$ are linear, which follows from equations 1.3.21 and 1.3.22.

Page 213 (print book; page 214 ebook) [added May 14, 2019] First note in the margin,lines 3–4: "first index of the t", not "first index of the p"

Page 213 [added May 14, 2019] In a number of places (line after Proposition and Definition 2.6.16, and in equation 2.6.21) all instances of $\Phi^{-1}_{\mathbf{w}}$ should be $\Phi^{-1}_{\{\mathbf{w}\}}$, and $\Phi_{\mathbf{v}}$ should be $\Phi_{\{\mathbf{v}\}}$.

Page 228 [added July 9, 2019] Right before the Perron-Frobenius theorem (Theorem 2.7.10) we have replaced the discussion of $A > \mathbf{0}$ with

- $A \geq B$ if all entries $a_{i,j}$ satisfy $a_{i,j} \geq b_{i,j}$
- $A > B$ if all entries $a_{i,j}$ satisfy $a_{i,j} > b_{i,j}$.

In particular, $A \geq \mathbf{0}$ if all $a_{i,j}$ satisfy $a_{i,j} \geq 0$, and $A > \mathbf{0}$ if all $a_{i,j}$ satisfy $a_{i,j} > 0$.

Page 229 [added March 25, 2017] Three lines after equation 2.7.40, we speak of $g(A\vec{\mathbf{v}})$, but g is defined on Δ, the set of unit vectors in Q, and there is no reason to think that $A\vec{\mathbf{v}}$ is a unit vector. To deal with this, we are replacing the first three paragraphs of the proof of Theorem 2.7.10 by:

Let $Q \subset \mathbb{R}^n$ be the "quadrant" $\vec{\mathbf{w}} \geq \vec{\mathbf{0}}$, set $Q^* \stackrel{\text{def}}{=} Q - \{\vec{\mathbf{0}}\}$, and let Δ be the set of unit vectors in Q. If $\vec{\mathbf{w}} \in \Delta$, then $\vec{\mathbf{w}} \geq \vec{\mathbf{0}}$ and $\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$, so (by Lemma 2.7.11) $A\vec{\mathbf{w}} > \vec{\mathbf{0}}$.

Consider the function $g : Q^* \to \mathbb{R}$ given by

$$g : \vec{\mathbf{w}} \mapsto \inf\left\{\frac{(A\vec{\mathbf{w}})_1}{w_1}, \frac{(A\vec{\mathbf{w}})_2}{w_2}, \dots, \frac{(A\vec{\mathbf{w}})_n}{w_n}\right\}; \qquad 2.7.39$$

then $g(\vec{\mathbf{w}})\vec{\mathbf{w}} \le A\vec{\mathbf{w}}$ for all $\vec{\mathbf{w}} \in Q^*$, and $g(\vec{\mathbf{w}})$ is the largest number for which this is true. Note that $g(\vec{\mathbf{w}}) = g(\vec{\mathbf{w}}/|\vec{\mathbf{w}}|)$ for all $\vec{\mathbf{w}} \in Q^*$.

Since g is an infimum of finitely many continuous functions $Q^* \to \mathbb{R}$, the function g is continuous. Since Δ is compact, g achieves its maximum on Δ at some $\vec{\mathbf{v}} \in \Delta$, which also achieves the maximum of g on Q^*. Let us see that $\vec{\mathbf{v}}$ is an eigenvector of A with eigenvalue $\lambda \stackrel{\text{def}}{=} g(\vec{\mathbf{v}})$. By contradiction, suppose that $g(\vec{\mathbf{v}})\vec{\mathbf{v}} \ne A\vec{\mathbf{v}}$. By Lemma 2.7.11, $g(\vec{\mathbf{v}})\mathbf{v} \le A\vec{\mathbf{v}}$ and $g(\vec{\mathbf{v}})\vec{\mathbf{v}} \ne A\vec{\mathbf{v}}$ imply

$$g(\vec{\mathbf{v}})A\vec{\mathbf{v}} = Ag(\vec{\mathbf{v}})\vec{\mathbf{v}} < AA\vec{\mathbf{v}}. \qquad 2.7.40$$

Since the inequality $g(\vec{\mathbf{v}})A\vec{\mathbf{v}} < AA\vec{\mathbf{v}}$ is strict, this contradicts the hypothesis that $\vec{\mathbf{v}}$ is an element of Q^* at which g achieves its maximum: $g(A\vec{\mathbf{v}})$ is the largest number such that $g(A\vec{\mathbf{v}})A\vec{\mathbf{v}} \le AA\vec{\mathbf{v}}$, so $g(A\vec{\mathbf{v}}) > g(\vec{\mathbf{v}})$.

Page 229 Two mistakes in the first margin note. In $A\vec{\mathbf{w}}' = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \end{bmatrix}$, the $\mathbf{w}'$ should be $\mathbf{w}$.

Three lines further down, $g : (\vec{\mathbf{w}})\vec{\mathbf{w}}$ should be $g(\vec{\mathbf{w}})\vec{\mathbf{w}}$.

In the second margin note, $A\mathbf{w}/w_i$ should be $(A\mathbf{w})_i/w_i$.

Page 229 [added May 13, 2017] In inequality 2.7.42: $A\big(g(\vec{\mathbf{w}})\vec{\mathbf{w}}\big)$, not $A\big(g(\vec{\mathbf{w}})\big)\vec{\mathbf{w}}$.

Page 231 [added May 14, 2019] The basis in Exercise 2.7.3, part b should be

$$p_1(x) = 1,\ p_2(x) = 1 + x,\ \dots,\ p_{k+1}(x) = 1 + x + \cdots + x^k.$$

Page 231 [added March 25, 2017] Exercise 2.7.4: The reference to Exercise 1.5.10 should begin this exercise, since the notation e^A is used in part a.

Page 242 [added May 22, 2017] In the second line of inequality 2.8.50, the term $4(u_2 - u_2)^2$ should be $4(u_2 - u_2')^2$.

Page 242 [added July 9, 2019] Nathaniel Schenker pointed out that in various places (page 252, for example) we used the notation $\vec{\mathbf{h}}_n$, only defined in Appendix A5. To deal with this issue, we have rewritten Theorem 2.8.13:

Theorem 2.8.13 (Kantorovich's theorem). *Let $\mathbf{a}_0$ be a point in $\mathbb{R}^n$, U an open neighborhood of $\mathbf{a}_0$ in $\mathbb{R}^n$, and $\vec{\mathbf{f}} : U \to \mathbb{R}^n$ a differentiable mapping whose derivative $[\mathbf{D}\vec{\mathbf{f}}(\mathbf{a}_0)]$ is invertible. Define*

$$\vec{\mathbf{h}}_0 \stackrel{\text{def}}{=} -[\mathbf{D}\vec{\mathbf{f}}(\mathbf{a}_0)]^{-1}\vec{\mathbf{f}}(\mathbf{a}_0), \quad \mathbf{a}_1 \stackrel{\text{def}}{=} \mathbf{a}_0 + \vec{\mathbf{h}}_0, \quad U_1 \stackrel{\text{def}}{=} B_{|\vec{\mathbf{h}}_0|}(\mathbf{a}_1). \qquad 2.8.51$$

If $\overline{U_1} \subset U$ and the derivative $[\mathbf{D}\vec{\mathbf{f}}]$ satisfies the Lipschitz condition

$$\big|[\mathbf{D}\vec{\mathbf{f}}(\mathbf{u}_1)] - [\mathbf{D}\vec{\mathbf{f}}(\mathbf{u}_2)]\big| \le M|\mathbf{u}_1 - \mathbf{u}_2| \quad \text{for all points } \mathbf{u}_1, \mathbf{u}_2 \in \overline{U_1}, \qquad 2.8.52$$

and if the inequality

$$\left|\vec{\mathbf{f}}(\mathbf{a}_0)\right| \left|[\mathbf{D}\vec{\mathbf{f}}(\mathbf{a}_0)]^{-1}\right|^2 M \le \frac{1}{2} \qquad 2.8.53$$

is satisfied, then the equation $\vec{\mathbf{f}}(\mathbf{x}) = \vec{\mathbf{0}}$ has a unique solution in $\overline{U_1}$: for all $i \ge 0$ we can define $\vec{\mathbf{h}}_i = -[\mathbf{D}\vec{\mathbf{f}}(\mathbf{a}_i)]^{-1}\vec{\mathbf{f}}(\mathbf{a}_i)$ and $\mathbf{a}_{i+1} = \mathbf{a}_i + \vec{\mathbf{h}}_i$, and the "Newton sequence" $i \mapsto \mathbf{a}_i$ converges to this unique solution.

PAGE 244 [added July 9, 2019] Proposition 2.8.14 is not correct as stated; one needs to add the condition that inequality 2.8.53 be satisfied.

PAGE 247 [added July 9, 2019] In three places, A_1 should be A': the line immediately before inequality 2.8.71, and in inequality 2.8.72 (two instances). The matrix A_1 appears, correctly, in the line after inequality 2.8.73 (and in the footnote); it is the matrix $A_1 = A_0 + H_0$.

PAGE 253 Three lines before inequality 2.9.6, "$\vec{\mathbf{h}}_n = |\mathbf{a}_{n+1} - \mathbf{a}_n|$" should be "$\vec{\mathbf{h}}_n = \mathbf{a}_{n+1} - \mathbf{a}_n$".

PAGE 274 [new, added Sept. 17 2020] Line immediately after equation 2.10.44: "$G(-2) = A_0$, not $G(0) = A_0$.

Exercise 2.23: Transposing the matrix $[\widetilde{A}^\top \,|\, \widetilde{B}^\top]$ gives $\begin{bmatrix} \widetilde{A} \\ \widetilde{B} \end{bmatrix}$; by "the columns of $\widetilde{B}$ corresponding to the zero columns of $\widetilde{A}$" we mean the columns of $\widetilde{B}$ that are then under the zero columns of $\widetilde{A}$, which form a block (perhaps empty) on the right of $\widetilde{A}$.

PAGES 274–275 [posted August 28, 2019] Last line of page 274:

$$\begin{pmatrix} A_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{should be} \quad \begin{pmatrix} A_0 \\ \operatorname{tr} A_0^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -2 \end{pmatrix}.$$

First displayed equation page 275: "i.e., for $t = 0$" should be "i.e., for $t = -2$."

PAGE 280 Exercise 2.23: In the last line, "the nonzero columns of $\widetilde{B}$ form a basis for the kernel of A" should be "the columns of $\widetilde{B}$ corresponding to the zero columns of $\widetilde{A}$ form a basis for the kernel of A". We are also changing the notation to reflect the notation used in the solution, so A is $n \times m$, not $k \times n$, and I_n becomes I_m.

PAGE 285 Five lines before Definition 3.1.2: "a smooth k-dimensional manifold", not "a smooth n-dimensional manifold".

PAGE 292 [added May 22, 2017] Last margin note: The displayed equation should have $D_n F(\mathbf{z})$, not $D_n(\mathbf{z})$.

PAGE 295 [new, added Sept. 17, 2020] Last line of the caption to Figure 3.1.12: "$y_2 - y_3$ is the same multiple of $x_2 - x_3$", not "$y_2 - y_3$ is a multiple of $x_2 - x_3$".

Page 307 [new], added Sept. 17, 2020] Line 3: Add C^1: "graph of a C^1 function".

Page 312 First line of proof of Proposition 3.2.11: "Choose $\mathbf{x} \in U$", not "Choose $\mathbf{x} \in \mathbf{g}(U)$".

Page 312 Paragraph following formula 3.2.33: The compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ both make sense, but it is $\Psi \circ \Phi$, not $\Phi \circ \Psi$, that takes U to U. The domain of the derivative $[\mathbf{D}\Psi((\mathbf{y}, \mathbf{0}))]$ is $\mathbb{R}^m \times T_{\mathbf{x}}M^{\perp}$, not $T_{\mathbf{x}}M \times T_{\mathbf{x}}M^{\perp}$. In any eventual reprinting, we will rewrite this paragraph as follows:

The derivative $[\mathbf{D}\Psi((\mathbf{y}, \mathbf{0}))]$ is the map $\mathbb{R}^m \times T_{\mathbf{x}}M^{\perp} \to \mathbb{R}^n$ that maps $[\vec{\mathbf{a}}, \vec{\mathbf{b}}]$ to $[\mathbf{D}\gamma(\mathbf{y})]\vec{\mathbf{a}} + \vec{\mathbf{b}}$. It is an isomorphism because $[\mathbf{D}\gamma(\mathbf{y})] : \mathbb{R}^m \to T_{\mathbf{x}}M$ is an isomorphism. So there is a neighborhood U of $\mathbf{x}$ in $\mathbb{R}^n$ and a C^1 map $\Phi : U \to V \times T_{\mathbf{x}}M^{\perp}$ such that $\Psi \circ \Phi = \mathrm{id} : U \to U$. The composition of Φ with the projection onto V is our desired extension.

Page 323 [added July 9, 2019] The proof of Theorem 3.3.16 did not include a proof of uniqueness. To remedy this, we have changed Proposition 3.3.17 to an "if and only if" statement, and then changed the proof of that proposition (in the appendix). The new Proposition 3.3.17 reads:

Proposition 3.3.17 (Size of a function with many vanishing partial derivatives). *Let U be an open subset of $\mathbb{R}^n$ and let $g : U \to \mathbb{R}$ be a C^k function. Then*

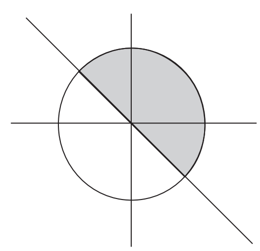
$$\lim_{\vec{\mathbf{h}} \to \vec{\mathbf{0}}} \frac{g(\mathbf{a} + \vec{\mathbf{h}})}{|\vec{\mathbf{h}}|^k} = 0 \qquad 3.3.34$$

if and only if all partial derivatives of g at $\mathbf{a} \in U$ up to order k vanish (including the 0th partial derivative, $g(\mathbf{a})$).

Page 365 [added August 1, 2017] Theorem 3.7.16: in two places in the third line, $\vec{\mathbf{v}}_i$ should be $\vec{\mathbf{v}}_1$:

. . . basis $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$ of $\mathbb{R}^n$ with $A\vec{\mathbf{v}}_i = \lambda_i \mathbf{v}_i$, where $\lambda_1, \dots, \lambda_k > 0$, . . .

Page 366 Exercise 3.7.10 refers to a picture that was not included. Here it is:



Page 369 Definition 3.8.4, part 3:

"If $A \cap B = \emptyset$ for $A, B \subset S$", not "If $\mathbf{P}(A \cap B) = \emptyset$ for $A, B \subset S$".

Page 396 Exercise 3.9.10, part c: The formula for $H(x)$ is missing a 1/2; it should be

$$H(x) = \frac{1}{2(1+(f'(x))^2)^{3/2}} \left(f''(x) - \frac{1+(f'(x))^2}{f(x)} \right).$$

Page 400 Exercise 3.29, part a: "then A' is still symmetric" should be "then B is symmetric".

Page 435 posted August 28, 2019] In the proof of Lemma 4.4.9, replace the first two sentences of the second paragraph by:

The sequence $j \mapsto \mathbf{x}_j$ is bounded, since the support of f is bounded and at least one of $\mathbf{x}_j$ and $\mathbf{y}_j$ is in the support of f, and they are close to each other. So it has a convergent subsequence $\mathbf{x}_{j_k}$ converging to some point $\mathbf{p}$.

Page 438 [added May 22, 2018] Exercise 4.4.6 is incorrect; the rationals in $[0, 1]$ are a counterexample. The exercise has been replaced by the following:

Show that if A$\subset \mathbb{R}^n$ has well-defined n-dimensional volume, then ∂A also has well-defined volume and $\text{vol}_n\, \partial A = 0$.

Page 471 [new, added Sept. 17, 2020] Line 7: "even number of permutations" should be "even number of transpositions".

Page 509 [added May 13, 2017] Second line: "by Theorem 4.11.4" should be "by part 3 of Proposition 4.1.14".

Page 513 [added, May 22, 2017] Equation 4.11.82: replace two dt by dx.

Page 528 [added May 22, 2018] Exercise 5.1.6 is incorrect as stated; vol_k is such a function V but it is not unique (for instance, the product of the lengths of the vectors is also such a function). The exercise and the margin note have been rewritten:

Exercise 5.1.6: We can think of V as a function of k vectors in $\mathbb{R}^n$: the columns of M. Multiplying by P on the left corresponds to rotating and reflecting these vectors (see Exercise 4.8.23). Multiplying by Q on the right is harder to visualize. It corresponds to rotating and reflecting the rows of M, i.e., the columns of $M^\top$.

5.1.6 Use the singular value decomposition (Theorem 3.8.1) to show that vol_k is the unique real-valued function V of $n \times k$ real matrices such that for all orthogonal $n \times n$ matrices P and all orthogonal $k \times k$ matrices Q, we have

$$V(M) = V(PMQ) \quad \text{and} \quad V(\sigma_1 \vec{\mathbf{e}}_1, \dots, \sigma_k \vec{\mathbf{e}}_k) = |\sigma_1 \cdots \sigma_k|$$

for any $n \times k$ real matrix M and any numbers $\sigma_1, \dots, \sigma_k$.

Page 572 [posted August 28, 2019] Line immediately before Example 6.1.11: "can be identified with $\mathbb{R}^3$", not "can be identified with vectors in $\mathbb{R}^3$".

Page 584 [added July 9, 2019] Line −2: "is either $\phi \mapsto +1$ or $\phi \mapsto -1$", not "is either $\{\phi\} \mapsto +1$ or $\{\phi\} \mapsto -1$".

PAGE 596 [new, added Sept. 17, 2020] Two errors in equation 6.4.28: at the end of the first line, $[\mathbf{D}\gamma_2(\mathbf{v})]\vec{\mathbf{e}}_k$ should be $[\mathbf{D}\gamma_2(\mathbf{u}_2)]\vec{\mathbf{e}}_k$. Similarly, at the beginning of the second line, $P_{\gamma_2(\mathbf{v})}$ should be $P_{\gamma_2(\mathbf{u}_2)}$.

PAGE 624 [added May 13, 2017] Exercise 6.6.2: By definition, a piece-with-boundary must be compact, but the loci in parts a and b are not compact. We are adding to each the additional constraint $x^2 + y^2 \leq 1$.

PAGE 639 [added May 13, 2017] Exercise 6.8.7: "For what vector field $\vec{F}$ can φ be written $W_{\vec{F}}$?", not "For what vector field $\vec{F}$ can φ be written $\mathbf{d}W_{\vec{F}}$?" (φ is a 1-form and $\mathbf{d}W_F$ is a 2-form, so the original wording did not make sense).

PAGE 653 [added May 13, 2017] Proposition 6.10.8: W is defined as a "bounded open subset of $\mathbb{R}^k$", but since W is open, it's not clear that it makes sense to "give $\partial(W \cap Z)$ the boundary orientation". Instead we should write "let W be a piece-with-boundary of $\mathbb{R}^k$", and in the 4th line, change "with compact support in W to "with compact support in the interior of W". Further, in the caption to Figure 6.10.6, " Since $\varphi = 0$ outside W" should be "Since $\varphi = 0$ on ∂W".

PAGE 710 [added July 9, 2019] At the top of the page, in several places, $i = 1, 2$ should be $i \in \{1, 2\}$, and $j = 1, 2, 3$ should be $j \in \{1, 2, 3\}$.

PAGE 717 Theorem 2.8.13 has been changed (see the erratum for page 242).

PAGE 727 First line after equation A7.10: "When $|\mathbf{y}_0 + \vec{\mathbf{k}}| \in V$" should be "When the point $\mathbf{y}_0 + \vec{\mathbf{k}}$ is in V".

PAGE 733 [added July 9, 2019] Proposition 3.3.17 has been changed to reflect the change page 323.

PAGE 734 [added July 9, 2019] In the original proof, now changed, some right parentheses are missing in equation A10.3.

PAGES 734-735 [added July 9, 2019] We have rewritten the proof of Proposition 3.3.17, which is now an "if and only if" statement. Here is the new proof: updater

Proof. First we show that if the partial derivatives vanish, the function is small. Without loss of generality we may assume $\mathbf{a} = \mathbf{0}$. We will prove the result by induction on k. The case $k = 0$ is one definition of continuity: if g is in $C^0(U)$, then $g(0) = 0$ if and only if $\lim_{h \to 0} g(h) = 0$.

Now suppose $k > 0$. Assume the theorem is true for $k - 1$. Suppose first that $D^I g(\mathbf{0}) = \mathbf{0}$ for all I with $|I| \leq k$. By the mean value theorem, there exists $c(\vec{\mathbf{h}}) \in (\mathbf{0}, \vec{\mathbf{h}})$ such that

$$g(\vec{\mathbf{h}}) = [\mathbf{D}g(c(\vec{\mathbf{h}}))]\vec{\mathbf{h}}. \qquad A10.1$$

Then

$$\lim_{\vec{\mathbf{h}}\to\mathbf{0}} \frac{|g(\vec{\mathbf{h}})|}{|\vec{\mathbf{h}}|^k} \overbrace{=}^{\text{Thm. 1.9.1}} \lim_{\vec{\mathbf{h}}\to\mathbf{0}} \frac{\left|[\mathbf{D}g(c(\vec{\mathbf{h}}))]\vec{\mathbf{h}}\right|}{|\vec{\mathbf{h}}|^k} \overbrace{\leq}^{\text{Prop. 1.4.11}} \lim_{\vec{\mathbf{h}}\to\mathbf{0}} \frac{|[\mathbf{D}g(c(\vec{\mathbf{h}}))]|}{|\vec{\mathbf{h}}|^{k-1}}$$

$$\underbrace{\leq}_{|c(\vec{\mathbf{h}})|\leq|\vec{\mathbf{h}}|} \lim_{\vec{\mathbf{h}}\to\mathbf{0}} \frac{|[\mathbf{D}g(c(\vec{\mathbf{h}}))]|}{|c(\vec{\mathbf{h}})|^{k-1}} \underbrace{=}_{\substack{\text{inductive}\\ \text{hypothesis}}} 0. \qquad A10.2$$

The last equality is the inductive hypothesis. This proves the first direction.

Now we show that if the function is small, i.e., if

$$\lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{g(\vec{\mathbf{h}})}{|\vec{\mathbf{h}}|^k} = 0, \qquad A10.3$$

then all the partial derivatives at the origin, up to order k, vanish.

We first prove this result in dimension 1. Let U be a neighborhood of 0 in $\mathbb{R}$ and g a function in $C^k(U)$. Assume that $\lim_{x\to 0} \frac{g(x)}{x^k} = 0$ (the function is small) and that

Recall that $f^{(k)}$ denotes the k derivative of f.

$$g(0) = g'(0) = \cdots = g^{(i-1)}(0) = 0 \quad \text{but} \quad g^{(i)}(0) \neq 0; \qquad A10.4$$

we will arrive at a contradiction. Below, to get the inequality going from the second line to the third line, note that since we are assuming $g^{(i)}(0) \neq 0$, we may assume (decreasing U and changing the sign of g if necessary) that there exists A such that $g^{(i)}(x) \geq A > 0$ for all $x \in U$:

$$\begin{aligned} g(x) &= \int_0^x g'(x_1)\,dx_1 = \int_0^x \int_0^{x_1} g''(x_2)\,dx_2\,dx_1 = \cdots \\ &= \int_0^x \int_0^{x_1} \cdots \int_0^{x_{i-1}} \underbrace{g^{(i)}(x_i)}\,dx_i\,dx_{i-1}\cdots dx_1 \\ &\geq \int_0^x \int_0^{x_1} \cdots \int_0^{x_{i-1}} \overbrace{A}\,dx_i\,dx_{i-1}\cdots dx_1 \qquad A10.5 \\ &= \int_0^x \int_0^{x_1} \cdots \int_0^{x_{i-2}} Ax_{i-1}\,dx_{i-1}\cdots dx_1 \\ &= \int_0^x \int_0^{x_1} \cdots \int_0^{x_{i-3}} \frac{Ax_{i-2}^2}{2}\,dx_{i-2}\cdots dx_1 = \cdots = \frac{Ax^i}{i!}. \end{aligned}$$

This gives the contradiction

$$\lim_{x\to\infty} \frac{g(x)}{|x|^k} \geq \lim_{x\to\infty} \frac{A}{i!}|x|^{i-k} \neq 0 \quad \text{if } i \leq k. \qquad A10.6$$

Now we prove that "function small implies derivatives 0" in dimension $n > 1$. For fixed $\mathbf{x} \in \mathbb{R}^n$, let $f_{\mathbf{x}}: \mathbb{R} \to \mathbb{R}$ be given by

$$f_{\mathbf{x}}(t) \overset{\text{def}}{=} g(t\mathbf{x}). \qquad A10.7$$

Since our hypothesis is that $g: U \to \mathbb{R}$ is small, $f_{\mathbf{x}}$ is small, so by the 1-dimensional case we have $f_{\mathbf{x}}^{(i)}(0) = 0$. Using the composition of Taylor polynomials, we can write

$$0 = f_{\mathbf{x}}^{(i)}(0) = \sum_{I \in \mathcal{I}_n^i} \frac{i!}{I!} \left(D_I g(\mathbf{0})\right) \mathbf{x}^I. \qquad \text{A10.8}$$

(Exercise A10.1 asks you to justify this equation.) The expression on the right is a polynomial function in n variables; this polynomial vanishes identically. Therefore, by equation 3.3.20, all its coefficients are 0: $D^I g(0) = 0$ for all multi-indices I with $\deg I \leq k$. □

PAGE 735 [added July 9, 2019] Exercise A10.1 has been replaced by

A10.1 Justify equation A10.8: let $U \subset \mathbb{R}^n$ be a neighborhood of $\mathbf{0}$, let $g : U \to \mathbb{R}$ be a function of class C^k, and let $\mathbf{x} \in \mathbb{R}^n$ be a point. Set $f_{\mathbf{x}}(t) = g(t\mathbf{x})$. Show that

$$f_{\mathbf{x}}^{(i)}(0) = \sum_{I \in \mathcal{I}_n^i} \frac{i!}{I!} D_I g(\mathbf{0}) \mathbf{x}^I.$$

PAGE 802 First line after A23.8: "if", not "if and only if":
this matrix is surjective if at least one . . .

Typos and minor errata

PAGE 25 Line 1: There should be no "s" between "consisting" and "alternately".

PAGE 27 [added July 9, 2019] Equation 0.7.8: the $\stackrel{\text{def}}{=}$ is misplaced; the equation should be

$$|z| = |a + ib| \stackrel{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{(a+ib)(a-ib)} = \sqrt{a^2 + b^2}. \qquad 0.7.8$$

PAGE 31 [added April 9, 2019] In Exercise 0.7.7, "all complex number" should be "all complex numbers".

PAGE 38 [added September 4, 2017] In the second line of the second paragraph, "because haven't defined" should be "because we haven't defined".

PAGE 51 [added Feb. 12, 2019] Figure 1.2.6: In the print version of the book, the line from V_1 to V_2 is too faint to show up.

PAGE 80 [added April 9, 2019] Line 7: "the three vector" should be "the three vectors".

PAGE 100 [added April 9, 2019] 3 lines before Proposition 1.5.38: k, not n: $(A_n)_{(i,j)}$ should be $(A_k)_{(i,j)}$.

PAGE 101 In the last line before Corollary 1.5.40, $(1 - A)$ should be $(I - A)$.

PAGE 107 [added April 9, 2019] In the caption for Figure 1.6.3, halfway through the first paragraph: m, not n: the $(m+1)$st digit, not the $(n+1)$st digit.

PAGE 109 [added May 27, 2017] Two lines before equation 1.6.9: Delete "is": "Next we want to show that f has a minimum."

PAGE 109 [added April 9, 2019] Line immediately before equation 1.6.9, add "in C: "There is a sequence $i \mapsto \mathbf{x}_i$ in C such that"

PAGE 113 [posted August 28, 2019] Third paragraph, two lines before end: "there is no complex number", not "there no complex number".

PAGE 119 [added April 9, 2019] Five lines from the bottom, there is a missing "on": "and so, up to a maximum" should be "and so on, up to a maximum".

PAGE 125 [added April 9, 2019] At the end of the caption for Figure 1.7.2, there is a hyphen that shouldn't be there.

PAGE 137 In the second line of Section 1.8, there should be no "s" after "given" (the extraneous s's no doubt come from an attempt to save the file).

PAGE 154 [added April 9, 2019] Exercise 1.13, part a: to avoid confusion, "side" should be "edge".

PAGE 160 [added May 14, 2019] Line 4: Perron-Frobenius theorem, not Perron-Frobenium theorem

PAGE 165 [new, added Sept. 17, 2020] Exercise 2.1.1, line 2 of part a: "format of equation 2.1.2", not "format of Exercise 1.2.2".

PAGE 174 [added May 14, 2019] Caption for matrix in margin: "Matrix for Exercise 2.2.11, discussion between parts c and d", not "Matrix for Exercise 2.2.11, part c"

PAGE 175 [new, added Sept. 17, 2020] The discussion immediately following Proposition 2.3.1 should have been labeled as a proof.

PAGE 176 [added May 14, 2019] Line 1 of Theorem 2.3.2: "an $n \times n$" not "a $n \times n$"

PAGE 185 [added May 14, 2019] Line 1: "with all 0's", not "with at all 0's"

PAGE 188 [new, added Sept. 17, 2020] The end of the proof of Proposition and Definition 2.4.19 is missing a period and an end-of-proof symbol.

PAGE 192 [added May 14, 2019] One line before Proposition 2.5.2: Section 0.4, not Section 0.3

PAGE 196 [new, added Sept. 17, 2020] In the second paragraph after Definition 2.5.9, $\mathbf{0}$ should be $\vec{\mathbf{0}}$.

PAGE 201 [new, added Sept. 17, 2020] The second margin note should end with a period.

PAGE 210 [added May 14, 2019] One line before Example 2.6.7: "is the same as", not "is same as"

PAGE 214 Second margin note, line 5: "subspace $E \subset \mathbb{R}^n$", not "subset $E \subset \mathbb{R}^n$".

PAGE 221 [new, added Sept. 17, 2020] Line immediately before Definition 2.7.3: λTV should be $\lambda T\mathbf{v}$.

PAGE 227 [added Sept. 30, 2019] Page 227: The last margin note should start "If A is a real matrix", not "If A s a real matrix".

PAGE 245 [added Sept. 17, 2020] In the second footnote, $\underbrace{\cos}_{\sin'} c$ should be $\underbrace{\cos c}_{\sin' c}$.

PAGE 253 [added August 28, 2019] Three lines before Theorem 2.9.4: $\mathbf{h}_n$ should be $\vec{\mathbf{h}}_n$.

PAGE 275 [added August 28, 2019] Line 4: "an implicit function", not "a implicit function".

PAGE 285 [new, added Sept. 17, 2020] In Definition 3.1.2., the words "smooth k-dimensional manifold" should be in italics.

PAGE 289 [new, added Sept. 17, 2020] Line 1: "no longer than" rather than "shorter". In the last margin note, change "the two angles above" to "two angles". The angles are described in the exercise.

PAGE 290 [new, added Sept. 17, 2020] Caption to Figure 3.1.9: "can't move", not "can't moved".

PAGE 296 [added July 4, 2017] Two lines before Theorem 3.1.16: "an arbitrary C^1 mapping", not "an arbitrary C^1 mappings".

PAGE 302 [new, added Sept. 17, 2020] Exercise 3.1.5: Replace "the set of equation $X_c = x^2 + y^3 = c$ by "the set X_c of equation $x^2 + y^3 = c$".

PAGE 311 Two lines before equation 3.2.31, the x in T_xM should be bold: $T_{\mathbf{x}}M$.

PAGE 311 [new, added Sept. 17, 2020] There should be a $\square$ at the bottom of the page to indicate the end of the proof.

PAGE 314 [new, added Sept. 17, 2020] Exercise 3.2.11, part d: $F^{-1}(0)$ should be $F^{-1}([0])$.

PAGE 314 [new, added Sept. 17, 2020] Second paragraph of the first margin note, problem with parentheses: "Part a, together with the associativity of matrix multiplication)" should be "Part a (together with the associativity of matrix multiplication)".

PAGE 314 [new, added Sept. 17, 2020] Section 3.3, line 4 of first paragraph: we will replace "the function and its derivative" by "the function and this linear approximation".

PAGE 316 Table 3.3.2: The 2x in the monomial row should be $2x$ (x in math mode)

PAGE 318 [added July 4, 2017] In the title of Theorem 3.3.8, the left parenthesis is missing.

PAGE 340 [added May 22, 2017] Three lines after equation 3.5.32: "identity" should be "identify".

PAGE 359 [added August 1, 2017] There is a misplaced end parenthesis in equation 3.7.39. The middle term should be $= [\mathbf{D}f(\gamma(\mathbf{v}_0))][\mathbf{D}\gamma(\mathbf{v}_0)] =$

PAGE 369 [new, added Sept. 17, 2020] In Definition 3.8.5, the words "random variable" should be in italics.

PAGE 376 [added May 22, 2017] Caption to Figure 3.8.6: in the third line, "face" should be "eigenface". In the last line, "egenvalue" should be "eigenvalue".

PAGE 404 [new, added Sept. 17, 2020] A period is missing at the end of Definition 4.1.3.

PAGE 407 [added August 28, 2019] Definition 4.1.12 should be "Definitions 4.1.12" and it should have the title "Integrable function, integral"; in the third line, "its integral" should be "its *integral*".

PAGE 420 [added August 1, 2017] Definition 4.2.6: "probability density", not "probability of density". "If f is", not "if f be".

Equation 4.2.13: The middle expression for the covariance is missing a set of parentheses. It should be

$$E\Big(\big(f - E(f)\big)\big(g - E(g)\big)\Big).$$

PAGE 427 Formula 4.3.8 is missing a closing parenthesis on the right: $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| < \epsilon$, not $|f(\mathbf{x}_1) - f(\mathbf{x}_2| < \epsilon$

PAGE 451 [added May 13, 2017] In Table 4.6.3, the word "Function" should be in the first row, not the second.

PAGE 488 [added Aug. 10, 2017] In the first margin note, to be consistent with equation 4.10.4, "the r in $r\,dr\,d\theta$ plays the role" should be "the r in $r\,|dr\,d\theta|$ plays the role".

PAGE 510 [added Aug. 12, 2017] In equation 4.11.67, on the right side, there is an extra end parenthesis: $(f \circ \Phi)(\mathbf{u})\big)$ should be $(f \circ \Phi)(\mathbf{u})$.

PAGE 526 [added July 9, 2019] 3 lines after subsection heading: "the area of " should be "the k-dimensional volume of".

PAGE 534 [added May 13, 2017] Line immediately before formula 5.2.15: we should have written $M \subset \mathbb{R}^n$, not just M.

PAGE 584 [added May 22, 2017] We forgot parentheses in the third line of Definition 6.3.3: $\mathcal{B}_{\mathbf{x}}(M)$, not $\mathcal{B}_{\mathbf{x}}M$.

PAGE 588 [added May 22, 2017] Exercise 6.3.13, part a: The first displayed equation is missing an arrow on $\mathbf{e}_1$.

PAGE 597 [added May 22, 2017] First line: "over the piece", not "through the piece". In the first line of equation 6.4.33, the two column vectors should be in parentheses:

$$P_{\begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix}}\left(\overbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}^{\overrightarrow{D_1\gamma}}, \overbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}^{\overrightarrow{D_2\gamma}}\right)$$

PAGE 599 [added May 13, 2017] The first margin note is missing an end parenthesis: "$\omega(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$" should be "$\omega(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k))$".

PAGE 612 [added May 22, 2017] Line 4: "require", not "rquire".

PAGE 614 [added May 13, 2017] Last line: "obviously", not "obvioiusly".

PAGE 693 [added July 9, 2019] Second line of equation 6.13.20: $|d\mathbf{t}^k|$ should be $|d^k\mathbf{t}|$

PAGE 705 First margin note: equation A1.2, not equation AA1.2

PAGE 719 Three lines before the bottom: "□ Lemma A5.3" indicating the end of the proof does not belong here; it goes after inequality A5.29. Inequality A5.26 should end with a period, not a comma.

PAGE 726 Inequality A7.8: the vertical line after $\mathbf{g}(\mathbf{y}_1)$ should be deleted; the first term should be $|\mathbf{g}(\mathbf{y}_1) - \mathbf{g}(\mathbf{y}_2)|$, not $|\mathbf{g}(\mathbf{y}_1)| - \mathbf{g}(\mathbf{y}_2)|$.

Page 733 [added May 22, 2017] Some standard basis vectors in equations A9.5, A9.6, and A9.7 are missing arrows: $\mathbf{e}_i$ should be $\vec{\mathbf{e}}_i$ and $\mathbf{e}_j$ should be $\vec{\mathbf{e}}_j$.

Page 811 [added May 22, 2017] The entry page 625 for the fundamental theorem of calculus should be 626.

Notes and clarifications

Pages 41-42 Exercise 1.1.8 could have been clearer. On page 41, "where a is" could be replaced by "the function a gives". On page 42, part b might include "The pipe forms a torus, with $r \leq 1$".

Page 75 [added August 28, 2019] Caption to Figure 1.4.9: this use of the word "span" is commonplace, but it is not the definition given in Definition 2.4.3.

Page 79 [added September 13, 2017] We prove part 3 of Proposition 1.4.20 for $\det[\vec{\mathbf{a}} \times \vec{\mathbf{b}}, \vec{\mathbf{a}}, \vec{\mathbf{b}}]$, but by the margin note on page 78, this is the same as proving it for $\det[\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{a}} \times \vec{\mathbf{b}}]$. Using the language of antisymmetry (see Theorem and Definition 4.8.1), going from one to the other requires two exchanges of arguments, so the sign changes twice and remains the same.

Page 79 In the proof of Proposition 1.4.21, θ is not well defined; there are two angles between a line l and a vector anchored at a point of l: one in $[0, \pi/2]$, the other in $[\pi/2, \pi]$. We want $\theta \in [0, \pi/2]$ to get $\cos\theta \geq 0$, so that the volume won't be negative.

Pages 91 and 92 [added September 4, 2017] We have added some comments about our definition of limits. At the bottom of page 91, next to equation 1.5.24, we are adding the margin note "If $\mathbf{x}_0$ is in the domain of f, one way to "approach" $\mathbf{x}_0$ is to *be* $\mathbf{x}_0$."

On page 92, in the margin, we have added some terminology. The standard U.S. definition is sometimes called the *deleted limit*; the definition we have adopted is sometimes called the *undeleted limit*.

Page 101 In the second printing we will replace the first sentence by

By Example 0.5.6, $|A| < 1$ implies that the geometric series $\sum_{k=0}^{\infty} |A|^k$ converges, so inequality 1.5.63 says that $\sum_{k=0}^{\infty} |A^k|$ converges if $|A| < 1$. Since absolute convergence implies convergence (Proposition 1.5.35), it follows that the series 1.5.61 defining S converges.

Page 110 [April 9, 2019] When we reprint the book we plan to add the following margin note next to equation 1.6.10:

"Here we use the $j \mapsto a_{i(j)}$ notation for subsequences introduced in Definition 1.5.18. In the proof of Theorem 1.6.11 (and in various places in Chapter 4 and the Appendix) we use the more standard notation a_{i_j}."

PAGES 113–116 [added July 9, 2019] We have made some changes to the proof of the fundamental theorem of algebra, in hopes that this will make it easier to follow. We are replacing the third and fourth paragraphs on page 113 by:

Part 1: Showing that $|p|$ has a minimum on $\mathbb{C}$

Our only criterion (very nearly *the* only criterion) for the existence of a minimum is Theorem 1.6.9, which requires that the domain be compact.[1] The domain of $z \mapsto |p(z)|$ is $\mathbb{C}$, which is not compact. But we will be able to show that there exists $R > 0$ such that if $|z| > R$ (i.e., z is *outside* the disc $\{ z \in \mathbb{C} \mid |z| \le R \}$), then $|p(z)| \ge |p(0)| = |a_0|$.

That will solve our problem, since Theorem 1.6.9 guarantees that there is a point z_0 in the disc at which p achieves its minimum: for all z with $|z| \le R$ we have $|p(z_0)| \le |p(z)|$, and in particular, $|p(z_0)| \le |p(0)| = |a_0|$.

But how can we choose R so that if z is outside the disc of radius R, then $|p(z)| \ge |a_0|$, so that the point z_0 in the disc is the minimum of $|p|$ on $\mathbb{C}$? Note that the function $1/(1+x^2)$ does not have a minimum on $\mathbb{R}$, and $|e^z|$ does not have a minimum on $\mathbb{C}$. In both cases, the infimum of the values is 0, but there is no $x_0 \in \mathbb{R}$ such that $1/(1+x_0^2) = 0$, and there no complex number z_0 such that $e^{z_0} = 0$.

On page 114 we have replaced the remark and the following three paragraphs by

Part 2: Showing that $p(z_0) = 0$

Now we need to show that z_0 is a root of the polynomial. The first part of the proof did not use the fact that p is complex valued. Now we must use that fact; real polynomials do not necessarily have real roots. We will argue by contradiction: we will assume $p(z_0) \ne 0$, and show that in that case there exists a point z such that $|p(z)| < |p(z_0)|$, contradicting our finding in the first part of the proof.

If p is a real polynomial, then $|p|$ achieves its minimum value at some x_0 with $|x_0| < R$, where R is constructed as in the proof. But x_0 is not necessarily a root; for example, for the polynomial x^2+1, the minimum value is 1, achieved at 0.

We will use the fact that when a complex number of length r and polar angle θ is written as $w = r(\cos\theta + i\sin\theta)$, then as θ goes from 0 to 2π, the point w turns in a circle of radius r around 0.

Our strategy will be to show that as z travels on a little circle around the minimum z_0 in the domain, the number $p(z)$ travels around $p(z_0)$; in doing so, it will come between $p(z_0)$ and the origin. When it does, $|p(z)|$ will be smaller than the proven minimum value $|p(z_0)|$, which is impossible.

It will be easier to consider numbers in a circle around z_0 if we treat z_0 as the origin, so we start with a change of variables. So set $z \stackrel{\text{def}}{=} z_0 + u$, and consider the function ...

On page 115 we are adding the margin note:

[1]Using Theorem 1.6.9 to prove existence of a minimum is what d'Alembert and Gauss did not know how to do; that is why their proofs were not rigorous. Topology was invented in large part to make these kinds of arguments possible.

For the first part of the proof, we used the property that for large z, the leading term z^k of a polynomial dominates the sum of all the other terms. Now we use the property that for small u, the lowest degree terms are the largest.

Page 132 [added May 27, 2017] First line of the footnote: We will replace "a vector H" by "the increment H; this "vector" is an $n \times n$ matrix." Vector spaces that are not naturally $\mathbb{R}^n$ are discussed in Section 2.6.

Page 141 [added April 9, 2019] In the second printing we will add the following margin note:

Recall that we use $[\mathbf{Df}(\mathbf{a})]$ to denote both the derivative of $\mathbf{f}$ at $\mathbf{a}$ and the matrix representing it. When we write equation 1.8.16 as a composition, we are thinking of the derivatives as linear transformations. In Figure 1.8.16 we write the composition as a matrix multiplication.

Page 170 Corollary 2.2.7 also uses Proposition 1.3.13; the proof of that proposition shows that if the linear transformation "multiplication by A" is bijective, its inverse mapping is linear and so has a matrix; this matrix is a two-sided inverse of A.

Page 171 [added July 9, 2018] We have rewritten (a second time) the paragraph following formula 2.2.11 was not clear: the antecedent of "it" in the second line was not specified, and in the next-to-last line we did not specify that $\vec{\mathbf{x}}' \neq \vec{\mathbf{x}}$. We propose instead:

> Next note that if B is a right inverse of A, i.e., $AB\vec{\mathbf{a}} = \vec{\mathbf{a}}$ for every $\vec{\mathbf{a}}$, then A is onto (since $A\vec{\mathbf{x}} = \vec{\mathbf{a}}$ has a solution for every $\vec{\mathbf{a}}$, namely $\vec{\mathbf{x}} = B\vec{\mathbf{a}}$) and B is one to one (since $B\vec{\mathbf{a}}_1 = B\vec{\mathbf{a}}_2$ implies $AB\vec{\mathbf{a}}_1 = AB\vec{\mathbf{a}}_2$, which implies $\vec{\mathbf{a}}_1 = \vec{\mathbf{a}}_2$). By formula 2.2.11, both A and B are one to one and onto. Since $AB\vec{\mathbf{a}} = \vec{\mathbf{a}}$, we have $BAB\vec{\mathbf{a}} = B\vec{\mathbf{a}}$ for all $\vec{\mathbf{a}}$, so $BA = I$ on the image of B. Since B is onto, the image of B is everything. So B is a left inverse of A.

We have also rewritten the second margin note, making it

> For any maps $f : X \to Y$ and $g : Y \to X$, if $g \circ f(x) = x$ for all $x \in X$, then f is 1–1 and g is onto. Moreover, g is a left inverse of f and f is a right inverse of g. But g is not necessarily a right inverse of f, and f is not necessarily a left inverse of g; in this setting, being 1–1 is not equivalent to being onto.

Page 214 [added September 15, 2017] At the end of the first margin note we plan to add:

Note that multiplying both sides on the right by $[P_{\mathbf{v}'\to\mathbf{v}}]^{-1}$ gives

$$[\mathbf{v}_1 \dots \mathbf{v}_n] = [\mathbf{v}_1' \dots \mathbf{v}_n'][P_{\mathbf{v}'\to\mathbf{v}}]^{-1};$$

the change of basis matrix from $\{\mathbf{v}'\}$ to $\{\mathbf{v}\}$ is the inverse of the change of basis matrix from $\{\mathbf{v}\}$ to $\{\mathbf{v}'\}$:

$$[P_{\mathbf{v}\to\mathbf{v}'}] = [P_{\mathbf{v}'\to\mathbf{v}}]^{-1}.$$

PAGE 214 [added May 13, 2019] We plan to add the following margin note:

If in Proposition and Definition 2.6.16 we set

$$V = W, \ \{\mathbf{v}\} = \{\mathbf{v}'\}, \ \{\mathbf{w}\} = \{\mathbf{v}\},$$

and let $T : V \to W$ be the identity, then we get Proposition and Definition 2.6.17: the change of basis matrix $[P_{\mathbf{v}' \to \mathbf{v}}]$ is the matrix of the identity with respect to the bases $\{\mathbf{v}\}$ and $\{\mathbf{v}'\}$.

PAGE 227 [added May 14, 2019] in any future printing we plan to move the margin note from the top of pager 228 and to modify it:

In equation 2.7.36, why can we use just one eigenvector per V_j? Each $\vec{\mathbf{w}}_{i,j}$ is a linear combination of eigenvectors of our eigenbasis. If $k < n$, then at least one V_j will satisfy $\dim V_j > 1$, and will contain $\dim V_j > 1$ elements of the chosen eigenbasis. The corresponding $\vec{\mathbf{w}}_{i,j}$ are linear combinations of these eigenvectors, and as such are themselves eigenvectors.

PAGE 229 [added March 22, 2017] We have added the following margin note, to address the issue why (2-3 lines after equation 2.7.41), "$\vec{\mathbf{v}}$ is the unique eigenvector of A in Δ":

"The plane P spanned by $\vec{\mathbf{v}}$ and $\vec{\mathbf{v}}'$ necessarily intersects ∂Q, and $A = \lambda \mathrm{id}$ on P means that $\partial Q \cap P$ is mapped to $\partial Q \cap P$, not to $\overset{\circ}{Q}$."

Note that the notation id for the identity transformation was introduced in Example 1.3.6.

PAGE 270 [added August 28, 2019] First line of the Remarks: "the k variables $\mathbf{b}$ determine the $n-k$ variables $\mathbf{a}$" should be "the k variables $\mathbf{y}$ corresponding to $\mathbf{b}$ determine the $n-k$ variables $\mathbf{x}$ corresponding to $\mathbf{a}$".

PAGE 275 [new, added Sept. 17, 2020] Exercise 2.10.1: In keeping with our usual practice, all the F should be $\mathbf{F}$. But other authors may use different conventions.

PAGE 277 [new, added Sept. 17, 2020] Exercise 2.10.15: as for Exercise 2.10.1, change F to $\mathbf{F}$.

PAGR 279 [new, added Sept. 17, 2020] Exercise 2.15 repeats Exercise 2.9, part c.

PAGE 282 [new, added Sept. 17, 2020] Exercise 2.37, part b: This should be stated as "if and only if":

Let p_1 and p_2 be polynomials of degree k_1 and k_2. Then p_1 and p_2 are relatively prime if and only if there exist unique polynomials q_1 and q_2 of degree at most $k_2 - 1$ and $k_1 - 1$ such that $p_1 q_1 + p_2 q_2 = 1$.

PAGE 296 [new, added Sept. 17, 2020] Proof of Theorem 3.1.16: That W is open follows from V being open and $\mathbf{f}$ continuous. Since $\mathbf{f}$ is continuous, if $\mathbf{y}$ is in W and $\mathbf{z}$ is a point in $\mathbb{R}^n$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\mathbf{z} - \mathbf{y}| < \delta \implies |\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{z})| < \epsilon$$

Since V is open, there exists $\epsilon > 0$ such that $B_\epsilon(\mathbf{f}(\mathbf{y}))$ is in V. Thus the ball of radius δ centered at $\mathbf{y}$ is a subset of W. So for every $\mathbf{y} \in W$, there exists δ such that $B_\delta(\mathbf{y})$ is a subset of W, so W is open.

Note for page 296: If V were not open, it would still be true that $W = \mathbf{f}^{-1}(V)$ would be open, but this requires a characterization of continuity not given in the book. To include it would require a definition of what it means for a subset V of a subset $X \subset \mathbb{R}^n$ to be open; our definitions of open and closed sets are limited to subsets of $\mathbb{R}^n$.

PAGE 311 Proposition and Definition 3.2.9: We used the notation for a restricted function before we explain it on page 359. Here, $[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]|_{T_\mathbf{x}M}$ denotes the linear function $[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})] : \mathbb{R}^n \to 277\mathbb{R}^k$ restricted to $T_\mathbf{x}M \subset \mathbb{R}^n$.

PAGE 311 Proof of Proposition 3.2.10: What justifies our saying (two lines before equation 3.2.31) that $\ker[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]$ and $T_\mathbf{x}M$ span $\mathbb{R}^n$? Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$; then $[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]\mathbf{u} \in \mathbb{R}^k$. Since $[\mathbf{D}\mathbf{f}(\mathbf{x})]$ is onto $\mathbb{R}^k$, there exists $\mathbf{u}_1 \in T_\mathbf{x}M$ such that

$$[\mathbf{D}\mathbf{f}(\mathbf{x})]\mathbf{u}_1 = [\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]\mathbf{u}.$$

By equation 3.2.27,

$$[\mathbf{D}\mathbf{f}(\mathbf{x})]\mathbf{u}_1 = [\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]\mathbf{u}_1,$$

so $[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]\mathbf{u} = [\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]\mathbf{u}_1$, i.e.,

$$\mathbf{u} - \mathbf{u}_1 \in \ker[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})].$$

Therefore we can write $\mathbf{u} \in \mathbb{R}^n$ as $\mathbf{u} = (\mathbf{u} - \mathbf{u}_1) + \mathbf{u}_1$, with $\mathbf{u}_1 \in T_\mathbf{x}M$ and $\mathbf{u} - \mathbf{u}_1 \in \ker[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]$.

PAGE 313 [new, added Sept. 17, 2020] Exercise 3.2.7: To keep notation consistent with that used in the solution, replace $\mathbf{x}_0$ by $\mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

PAGE 317 [new, added Sept. 17, 2020] Last line of last margin note: Corollary 3.3.10 would be a better reference.

PAGE 319 [added July 9, 2019] In the second printing we will add a margin note: It follows from Proposition 3.3.11 that if a polynomial vanishes identically (so that its derivatives are all 0), its coefficients are all 0.

PAGE 347 [added July 9, 2019] Theorem 3.6.10 is contained in Theorem 3.6.8; the separate proof on page 347 is not necessary.

PAGE 352 Theorem and Definition 3.7.5: We neglected to specify how the C^1 mapping $\mathbf{F}$ defines X; it defines it by the equation

$$X \stackrel{\text{def}}{=} \mathbf{F}^{-1}(\mathbf{0}).$$

PAGE 402 We have elaborated on the notion (beginning of third paragraph) of "easiest to prove":

Defining volume using dyadic pavings as we do in Section 4.1, gives the easiest way to prove most theorems, since all dyadic pavings are comparable: for any two dyadic pavings, one is a refinement of the other: as we cut the domain into smaller and smaller cubes, the smaller cubes fit exactly into the larger cubes. This is especially important in higher dimensions.

PAGE 451 [added May 13, 2017] We are adding as a margin note: Proof of part 1: The change of variables

$$x \mapsto \frac{1}{2}\big((1+x)b + (1-x)a\big)$$

will bring an integral over any segment $[a, b]$ to an integral over $[-1, 1]$; when integrating a polynomial this does not change the degree of the polynomial.

PAGE 461 [added May 22, 2017] First paragraph of Section 4.8: It would probably be better to write "gives the signed area" and "gives the signed volume", and to replace "geometric interpretation, as a signed volume, by "geometric interpretation as a signed n-dimensional volume".

PAGE 502 [added Sept. 14, 2017] We have changed definition 4.11.8 to include an explicit definition of "Lebesgue-integrable function":

Definition 4.11.8 (Lebesgue-integrable function, Lebesgue integral). Let $k \mapsto f_k$ be a sequence of R-integrable functions such that

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\mathbf{x})||d^n\mathbf{x}| < \infty. \qquad 4.11.19$$

Then $f \underset{L}{=} \sum_{k=1}^{\infty} f_k$ is *Lebesgue integrable*, and its *Lebesgue integral* is

$$\int_{\mathbb{R}^n} f(\mathbf{x})|d^n\mathbf{x}| \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(\mathbf{x})|d^n\mathbf{x}|. \qquad 4.11.20$$

PAGE 509 [added August 28, 2019] The last margin note should be: "Theorem 4.11.17: If the f_k are Riemann-integrable and inequality 4.11.62 is satisfied, this is simply the definition of the Lebesgue integral."

PAGE 527 [added May 22, 2017] The notion of a vector anchored at a point is discussed in Section 1.1. We might spell out what is meant here: "We denote by $P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ a k-parallelogram in $\mathbb{R}^n$ anchored at $\mathbf{x} \in \mathbb{R}^n$: the k vectors spanning the parallelogram all begin at $\mathbf{x}$."

PAGE 529 [added May 13, 2017] Definition 5.2.3: We are expanding part 4, which is now “the derivative $[D\gamma(\mathbf{u})]$ is one to one for all $\mathbf{u}$ in $U - X$, with image $T_{\gamma(\mathbf{u})}M$”

We are also adding a margin note:

“Part 4: Recall from Proposition 3.2.7 that $T_{\gamma(\mathbf{u})}M = \operatorname{img}[\mathbf{D}\gamma(\mathbf{u})]$.”

PAGE 570 [added August 28, 2019] In the second printing, in Definition 6.1.7 (and in subsequent places) we are replacing the notation $A_c^k(\mathbb{R}^n)$ by $A_{const}^k(\mathbb{R}^n)$. The new notation is heavy, but it avoids confusion with a fairly standard use where “ sub c” denotes “compact”.

PAGE 575 [added May 22, 2017] Equation 6.1.40 has $\mathbf{x}$ on the right, no $\mathbf{x}$ on the left. We will take out the $\mathbf{x}$ on the right and elaborate, writing

$$\varphi = \sum_{1\le i_1<\cdots<i_k\le n} a_{i_1,\dots,i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}; \qquad 6.1.40$$

for each $i_1, \dots, i_k$ satisfying $1 \le i_1 < \cdots < i_k \le n$, the map $\mathbf{x} \mapsto a_{i_1,\dots,i_k}(\mathbf{x})$ is a real-valued function on U.

PAGE 584 [added August 28, 2019] We have added the notation Ω and $\Omega_{\mathbf{x}}$ to Definition 6.3.3:

Definition 6.3.3. An *orientation* of a k-dimensional manifold $M \subset \mathbb{R}^n$ is a continuous map $\Omega : \mathcal{B}(M) \to \{+1, -1\}$ whose restriction $\Omega_{\mathbf{x}}$ to each $\mathcal{B}_{\mathbf{x}}(M)$ is an orientation of $T_{\mathbf{x}}M$.

PAGES 585–586 [added August 28, 2019] In equation 6.3.9 and in the line following it, $\Omega^{\vec{\mathbf{n}}}$ should be $\Omega^{\vec{\mathbf{n}}}_{\mathbf{x}}$. In equation 6.3.15 Ω should be $\Omega_{\mathbf{x}}$.

PAGE 585 [added May 22, 2017] At the end of Example 6.3.6, we will add a reference to concentric circles: “ . . . is a nonvanishing vector field tangent to the circle of equation $x^2+y^2 = R^2$ (and to all concentric circles), defining the *counterclockwise* orientation.”

PAGE 587 [added May 13, 2017] Proof of Proposition 6.3.10: We are replacing the part starting with “Choose a path” by the following (Exercise 6.3.16 is new; see below):

Choose a continuous path $\gamma : [a, b] \to M$ with $\gamma(a) = \mathbf{x}_0$ and $\gamma(b) = \mathbf{x}$. For all t there exists $s(t) \in \{+1, -1\}$ such that $\Omega'_{\gamma(t)} = s(t)\Omega''_{\gamma(t)}$. If s is not constant, there exists a sequence $t_n \to t_\infty$ in $[a, b]$ such that $s(t_n) = -1$ for all n, but $s(t_\infty) = +1$. Choose a basis $\{\mathbf{b}\}$ of $T_{\gamma(t_\infty)}M$ such that $\Omega'(\gamma(t_\infty), \{\mathbf{b}\}) = \Omega''(\gamma(t_\infty), \{\mathbf{b}\})$. By Exercise 6.3.16 there exist bases $\{\mathbf{b}_n\}$ of $T_{\gamma(t_n)}M$ such that $(\gamma(t_n), \{\mathbf{b}_n\})$ converges to $(\gamma(t_\infty), \{\mathbf{b}\})$. Then

$$s(t_n) = \frac{\Omega'(\gamma(t_n), \{\mathbf{b}_n\})}{\Omega''(\gamma(t_n), \{\mathbf{b}_n\})} = -1 \quad \text{but} \quad \frac{\Omega'(\gamma(t_\infty), \{\mathbf{b}\})}{\Omega''(\gamma(t_\infty), \{\mathbf{b}\})} = +1.$$

But Ω'/Ω'' is continuous (Theorem 1.5.29), so s is constant. □

PAGE 589 [added May 13, 2017] Exercise 6.3.16 is new, as is the hint in the margin.

Hint for Exercise 6.3.16: Denote by $\mathrm{GL_k}\mathbb{R}$ the set of invertible $k \times k$ real matrices (which is open in $\mathrm{Mat}\,(n,n)$ by Corollary1.5.40). Suppose that U is a subset of $\mathbb{R}^k$ and $\gamma : U \to \mathbb{R}^n$ is a parametrization of an open subset of M. Show that the map

$$\Gamma : U \times \mathrm{GL_k}\,\mathbb{R} \to \mathbb{R}^{n(k+1)}$$

defined by

$$\Gamma(\mathbf{u}, [\mathbf{a}_1, \dots, \mathbf{a}_k] = \Big(\gamma(\mathbf{u}), [\mathbf{D}\gamma(\mathbf{u})]\mathbf{a}_1, \dots [\mathbf{D}\gamma(\mathbf{u})]\mathbf{a}_k\Big)$$

parametrizes an open subset of $\mathcal{B}(M)$.

6.3.16 Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold in $\mathbb{R}^n$. Prove that $\mathcal{B}(M)$ is a manifold in $\mathbb{R}^{n(k+1)}$, and that every open subset of $\mathcal{B}(M)$ projects to an open subset of M. See the hint in the margin.

PAGE 727 [added July 9, 2019] We have rewritten the paragraph starting "Let us see that $\mathbf{g}(V)$ is open":

Let us see that $\mathbf{g}(V)$ is open. From the argument above, if $\mathbf{g}(V)$ contains a neighborhood of $\mathbf{x}_0$, then it will contain a neighborhood of any point of $\mathbf{g}(V)$. The set $\mathbf{g}(V)$ does contain a neighborhood of $\mathbf{x}_0$: if $\mathbf{x}$ is sufficiently close to $\mathbf{x}_0$ then, since $\mathbf{f}$ is continuous, $\mathbf{f}(\mathbf{x})$ is in V, so $\mathbf{x}_1 \overset{\text{def}}{=} \mathbf{g}(\mathbf{f}(\mathbf{x}))$ is in $\mathbf{g}(V)$. Since

$$\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{f}(\mathbf{x}), \qquad A7.9$$

the injectivity of $\mathbf{f}$ implies $\mathbf{x} = \mathbf{x}_1$, so $\mathbf{x}$ is in $\mathbf{g}(V)$. Thus $\mathbf{g}(V)$ is open.