Vector Calculus, Linear Algebra and Differential Forms: A Unified Approach
5th Edition, Second Printing

Complete List of Errata and Notes as of September 15, 2022

You have the first printing if the copyright page contains, after "Printed in the United States of America" the numbers 10 9 8 7 6 5 4 3 2 1.

You have the second printing if the numbers are 10 9 8 7 6 5 4 3 2 (ending with 2).

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The list is divided into three sections: errata; minor typos; and notes and clarifications. New errata are marked in red.

Errata

Page 114 [added Oct. 26, 2020] Line 1: $|p|$, not p: "Therefore $|p|$ has a global minimum on $\mathbb{C}$" (see the first margin note on page 113).

Page 116 [added Jan. 6, 2022] Line 4: The left side of De Moivre's formula is missing an open parenthesis; it should be $\big(r(\cos\theta + i\sin\theta)\big)^k$.

Page 146 [added Jan. 6, 2022] Margin note: $\mathbf{c}$ should be $\mathbf{c}_0$:

$$|f(\mathbf{b}) - f(\mathbf{a})| \le |[\mathbf{D}f(\mathbf{c}_0)]|\ |\overrightarrow{\mathbf{b}-\mathbf{a}}|.$$

Page 177 [added Jan. 6, 2022] Three lines from the bottom: "a product of elementary matrices", not "a product of invertible matrices".

Page 271 [added Jan. 6, 2022] Part 2 of the remark: W_0 is not quite correct. We are replacing the first two sentences of this paragraph by

The proof of the implicit function theorem given in Appendix A8 shows that in an appropriate neighborhood Z of $\mathbf{c}$, every solution to $\mathbf{F}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \mathbf{0}$ is of the form $\begin{pmatrix}\mathbf{g}(\mathbf{y})\\ \mathbf{y}\end{pmatrix}$. Thus the intersection of Z with the graph of $\mathbf{g}$ is the intersection of Z with the solution set of $\mathbf{F}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \mathbf{0}$.

In the appendix, the neighborhood Z is called $\widetilde{G}\left(B_R\begin{pmatrix}\mathbf{0}\\ \mathbf{b}\end{pmatrix}\right)$.

Page 295 Last line of the caption to Figure 3.1.12: "$y_2 - y_3$ is the same multiple of $x_2 - x_3$", not "$y_2 - y_3$ is a multiple of $x_2 - x_3$".

PAGE 307 Line 3: Add C^1: "graph of a C^1 function".

PAGE 338 [added Jan. 6, 2022] Corollary 3.5.12: We should have said that A is linear.

PAGE 348 [added 8/7/21] Exercise 3.6.3 should be deleted; the proof is complete.

PAGES 351–352 [added 8/7/21] Example 3.7.4: To be consistent with the arrows in the figure, in the 4th line, $x_1 = x_3$ should be $x_1 + x_3 = 0$, and in equation 3.7.6, the second line on the right should be $x_1 + x_3$. In equation 3.7.8, the entry -1 in $[\mathbf{DF}(\mathbf{a})]$ should be 1. To be consistent, in Figure 3.7.3 and in the last two lines of the example, the notation x, y, z should be x_1, x_2, x_3. In the next-to-last line of the example, $x - z = 0$ should be $x_1 + x_3 = 0$.

PAGE 358 [added 8/7/21] First line of proof of Theorem 3.7.5: the reference should be to equation 3.7.1, not equation 3.7.11.

PAGE 359 [added 8/7/21] 4th paragraph, margin note about Lemma 3.7.11: $m \leq n$ not $k \leq n$. But we are changing this note and moving it to page 358, next to Lemma 3.7.11. The rewritten note reads:

Lemma 3.7.11 is true for any n and m, but when applying it to equation 3.7.1 we have $m \leq n$, since in that case, A is $[\mathbf{DF}]$, which is required to be onto.

PAGE 371 [added Jan. 6, 2022] Part 1: in the displayed equation, f_1, f_m, and f_i should all be f.

PAGE 376 [added Jan. 6, 2022] In equation 3.8.32 and the first half of the next line, we need transposes:

$$C^\top = [(p_1 - \overline{p})^\top, \dots, (p_m - \overline{p})^\top]; \qquad 3.8.32$$

i.e., the ith column of $C^\top$ is $(p_i - \overline{p})^\top$.

PAGE 415 [added Sept. 17, 2020] Exercise 4.1.5, part d: The sentence "The first is shown in the figure in the margin" belongs with part c.

PAGE 416 [added Jan. 6, 2022] Exercise 4.1.16, parts a and b: S should be Q.

PAGE 452 [added Jan. 6, 2022] Equation 4.6.7: $f(x_i)$ should be $p(x_i)$.

PAGE 455 [added Jan. 6, 2022] Margin note starting "Equation 4.6.20": the b_i at the end of the first paragraph should be $\sum a_i$ and $\sum a_i^2$.

PAGE 458 [new] Exercise 4.6.6, part c: the line immediately before the displayed equation should specify "for all x": Show that that for all x there exists $c \in (a_0, a_n)$ such that

$$f(x) - p(x) = \frac{f^{(n+1)}(c)}{(n+1)!} q(x).$$

PAGE 471 [added Sept. 17, 2020] Line 7: "even number of permutations" should be "even number of transpositions".

PAGE 504 [added Jan. 6, 2022] Page 504: In equation 4.11.29, the $\mathbf{x}$ should be moved outside the absolute values: $\big|[H_l]_R - H_M\big|(\mathbf{x})$

PAGE 513 [added Oct. 26, 2020] Last line of Example 4.11.23: In two places, f should be F.

PAGE 516 [added Jan. 6, 2022] First margin note: $d\xi$ not dx in the displayed equation.

PAGE 522 [added Jan. 6, 2022] The lemma in Exercise 4.26, part d requires an additional hypothesis: the C_n tend to 0.

PAGE 529 [new] Definition 5.2.3: in part 3, we are adding the condition that γ have locally Lipschitz derivative. What this means is explained in Proposition and Defintion 3.2.9, where $\mathbf{f}$ plays the role of γ^{-1} here. See the new entry for page 311, under "notes and clarifications".

PAGE 530 [added Jan. 6, 2022] Example 5.2.4: In the paragraph beginning "Condition 5 is satisfied", the two sentences beginning "Moreover" can be deleted: the lines $\gamma\begin{pmatrix}0\\\theta\end{pmatrix}$, $\gamma\begin{pmatrix}1\\\theta\end{pmatrix}$, $\gamma\begin{pmatrix}r\\0\end{pmatrix}$, and $\gamma\begin{pmatrix}r\\2\pi\end{pmatrix}$ are not in M, so their intersection with a compact subset $C \subset M$ is the empty set.

PAGE 535 [new] In Theorem 5.2.11, "Lipschitz inverse" should be "Lipschitz derivative". In Lemma 5.2.12, $B_r(\gamma_1(\mathbf{y})) \subset \gamma_1(B_{Lr}(\mathbf{y}))$ should be $\mathrm{B}_r(\gamma_1(\mathbf{y})) \cap M \subset \gamma_1(B_{Lr}(\mathbf{y}))$. (The subset $B_r(\gamma_1(\mathbf{y}))$ is a subset of $\mathbb{R}^n$, the ambient space of M, while $\gamma_1(B_{Lr}(\mathbf{y}))$ is a subset of M.)

PAGE 555 [new] Formula 5.4.26 is no longer displayed, but we are leaving the subsequent equation numbers unchanged,

PAGE 559 [new] Exercise 5.4.5: the definition of mean curvature should include a factor of $1/n$: $H(\mathbf{x}_0) \stackrel{\text{def}}{=} \frac{1}{n}(a_1 + \cdots + a_n)$.

PAGE 560 [new] Exercise 5.4.7: because of the change in the definition of H (see the entry for page 559), the Taylor polynomial for $\mathrm{vol}_n(M_t)$ should be

$$\mathrm{vol}_n(M_t) = \mathrm{vol}_n M - nt \int_M \vec{H}(\mathbf{x}) \cdot \vec{\mathbf{w}}(\mathbf{x}) |d^n\mathbf{x}| + o(t),$$

(Note also that the integral is over M, not S.)

Page 572 [added Oct. 26, 2020] In Definition 6.1.12, the summation should be over shuffles $\sigma \in \text{Perm}(k+l)$, not $\text{Perm}(k,l)$.

Page 589 [added Jan. 6, 2022] Exercise 6.3.16: we should have said that the manifold M is of class C^2. We are also deleting the hint for the exercise, since using it would require knowing that a set that can be parametrized is a manifold.

Page 592 [new] Proposition 6.4.6: In the last sentence, U should be $U-X$ (in two places). In Example 6.4.7 we are changing the sentence that starts "Does the parametrization":

Does the parametrization

$$\gamma\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (R+r\cos u)\cos v \\ (R+r\cos u)\sin v \\ r\sin u \end{pmatrix}, \quad \text{for } U = [-\pi,\pi]\times[-\pi,\pi], \qquad 6.4.11$$

with $X = \partial U$, preserve that orientation on $U-X$?

Page 596 [added Sept. 17, 2020] Two errors in equation 6.4.28: at the end of the second line, $[\mathbf{D}\gamma_2(\mathbf{v})]\vec{\mathbf{e}}_k$ should be $[\mathbf{D}\gamma_2(\mathbf{u}_2)]\vec{\mathbf{e}}_k$. Similarly, at the beginning of the third line, $P_{\gamma_2(\mathbf{v})}$ should be $P_{\gamma_2(\mathbf{u}_2)}$.

Page 596 [added 8/7/21] Arrows are missing two lines before equation 6.4.29: "corresponds to $\det[\mathbf{D}\Phi(\mathbf{u}_1)] = \det[D_1\Phi(\mathbf{u}_1),\dots,D_k\Phi(\mathbf{u}_1)]$" should be "corresponds to $\det[\mathbf{D}\Phi(\mathbf{u}_1)] = \det[\overrightarrow{D_1\Phi}(\mathbf{u}_1),\dots,\overrightarrow{D_k\Phi}(\mathbf{u}_1)]$".

Page 615 [added 8/7/21] Caption for Figure 6.6.7, line 7: "consists of three points", not "consists of a single point".

Page 618 [added 8/7/21] Proof of Proposition 6.6.18: We should have stated at the beginning that we were first proving the proposition in the case where A is orthogonal.

Page 629 [new] Caption to Figure 6.7.1: for the exercise to make sense, we should have specified orientation: "Exercise 6.7.12 asks you to show that if the projection preserves orientation, the integral ... "

Page 630 [new] Theorem 6.7.8: "class C^2" should be "class C^3", since Theorem 6.7.4 requires class C^2. Theorem 6.7.8 is probably true for C^2, but it is definitely not true for C^1, since the proof requires equality of crossed partials.

Page 638 [new] Two lines before equation 6.8.12: divided by, not times: "returns approximately the integral of φ over the box's boundary divided by (scaled by) the volume of the box."

Page 645 [added 8/7/21] Exercise 6.9.3: since f is vector valued, we are changing f to $\mathbf{f}$ throughout. This is not an error, but there are errors in both the statement of the exercise and in the solution: in several places we write $\xi(\mathbf{x})$, which does not make sense, since ξ is a vector field on the codomain of $\mathbf{f}$ ($\mathbb{R}^m$ in part a) but $\mathbf{x}$ is in the domain $\mathbb{R}^n$. All the $\xi(\mathbf{x})$ should be $\xi(\mathbf{f}(\mathbf{x}))$.

PAGE 655 [new] Equation 6.10.48: on the right side, the integral is over $\partial(W \cap Z)$.

PAGE 658 [new] Equations 6.10.66, second equation: the $=$ should be $\underset{L}{=}$. The proof is about Lebesgue integrals, because it involves exchanging limits and integrals.

PAGE 661 [new] Exercise 6.10.11: the function F is not defined at $\mathbf{0}$. The line before equation 1 should read "Define $F : (U - \{\mathbf{0}\}) \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by" and part a should read

a. Show that F extends to a C^2 map $\widetilde{F}$ on $U \times \mathbb{R}^{n-k}$ and that $\left[\mathbf{D}\widetilde{F}\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}\right]$ is the identity.

In part b, F should be $\widetilde{F}$.

PAGE 666 [new] In the remark, "dividing by the square root of the length" should be "dividing by the length, which involves a square root".

PAGES 689 AND 690 [new] Two lines after equation 6.13.5: ($\mathbb{R}^3 - z$-axis) should be ($\mathbb{R}^3 - x$-axis). In the paragraph after Example 6.13.3, z-axis should be x-axis (two places).

PAGR 731 [added Jan. 6, 2022] First line of the remark: "a point $\begin{pmatrix}\mathbf{g}(\mathbf{y})\\ \mathbf{y}\end{pmatrix} \in B_R\begin{pmatrix}\mathbf{0}\\ \mathbf{b}\end{pmatrix}$," should be "a point $\begin{pmatrix}\mathbf{g}(\mathbf{y})\\ \mathbf{y}\end{pmatrix}$ with $\mathbf{y} \in B_R(\mathbf{b})$".

PAGE 743 [added Jan. 6, 2022] Equation A12.12: on the left, $P^k_{f,\mathbf{a}}(0+1)$ should be $P^k_{g,0}(0+1)$. Equation A12.13: on the left of the first line (t) should be $(t+c)$. There are two instances of this.

PAGE 744 [added Jan. 6, 2022] Equation A12.18: the right side is missing a $1/(k+1)!$.

PAGES 749–750 [added Jan. 6, 2022] Siqi Wang found a counterexample to part 1 of Proposition A14.1; the second entry in the first row and first entry in the second row are not necessarily 0. Fortunately part 2 (the part we need) is still true, but the statement and proof need to be rewritten, as follows:

Proposition A14.1.

1. *The* $(n+2m)\times(n+2m)$ *Hessian matrix of* $L_{\widetilde{f},\widetilde{\mathbf{F}}}$ *at* $\begin{pmatrix}\mathbf{x}_0\\ \mathbf{0}\\ \boldsymbol{\lambda}_0^\top\end{pmatrix}$ *is*

$$\begin{bmatrix}[D_\mathbf{x}D_\mathbf{x}\widetilde{L}] & [D_\mathbf{x}D_\mathbf{u}\widetilde{L}] & [D_\mathbf{x}D_{\boldsymbol{\lambda}}\widetilde{L}]\\ [D_\mathbf{u}D_\mathbf{x}\widetilde{L}] & [D_\mathbf{u}D_\mathbf{u}\widetilde{L}] & [D_\mathbf{u}D_{\boldsymbol{\lambda}}\widetilde{L}]\\ [D_{\boldsymbol{\lambda}}D_\mathbf{x}\widetilde{L}] & [D_{\boldsymbol{\lambda}}D_\mathbf{u}\widetilde{L}] & [D_{\boldsymbol{\lambda}}D_{\boldsymbol{\lambda}}\widetilde{L}]\end{bmatrix} = \begin{bmatrix}[D^2_\mathbf{x}\widetilde{f}] & [D_\mathbf{x}D_\mathbf{u}\widetilde{f}] & [0]\\ [D_\mathbf{u}D_\mathbf{x}\widetilde{f}] & [D^2_\mathbf{u}\widetilde{f}] & -I_m\\ [0] & -I_m & [0]\end{bmatrix},$$

where $\widetilde{L} \overset{\text{def}}{=} L_{\widetilde{f},\widetilde{\mathbf{F}}} : \mathbb{R}^{n+2m} \to \mathbb{R}$.

2. *The corresponding quadratic form has signature* $(p+m, q+m)$.

Proof. 1. The Hessian matrix in Proposition A14.1 has nine submatrices; by symmetry, only six need to be examined:

$[D_\mathbf{x}D_\mathbf{x}\widetilde{L}]$ We have $[D_\mathbf{x}D_\mathbf{x}\widetilde{L}] = [D^2_\mathbf{x}\widetilde{f}]$, since there are no $\mathbf{x}$-terms in $\boldsymbol{\lambda}\mathbf{u}$.

$[D_\mathbf{u}D_\mathbf{u}\widetilde{L}]$ The $\mathbf{u}$ variables only appear to degree 1 in $\boldsymbol{\lambda}\mathbf{u}$, so the second derivatives with respect to $\mathbf{u}$ vanish, giving $[D_\mathbf{u}D_\mathbf{u}\widetilde{L}] = [D^2_\mathbf{u}\widetilde{f}]$.

$[D_\mathbf{u}D_{\boldsymbol{\lambda}}\widetilde{L}]$ The term $\widetilde{f}$ does not contribute, since it contains no $\boldsymbol{\lambda}$ terms; the second derivatives of $-\boldsymbol{\lambda}\mathbf{u}$ form $-I_m$, the $m\times m$ identity matrix.

$[D_{\boldsymbol{\lambda}}D_{\boldsymbol{\lambda}}\widetilde{L}]$ The second derivatives of $L_{\widetilde{f},\widetilde{\mathbf{F}}}$ with respect to the $\boldsymbol{\lambda}$ variables vanish.

$[D_\mathbf{x}D_{\boldsymbol{\lambda}}\widetilde{L}]$ The derivatives first with respect to $\boldsymbol{\lambda}$-variables and then with respect to the $\mathbf{x}$-variables vanish.

$[D_\mathbf{u}D_\mathbf{x}\widetilde{L}]$ We have $[D_\mathbf{u}D_\mathbf{x}\widetilde{L}] = [D_\mathbf{x}D_\mathbf{u}\widetilde{L}]^\top = [D_\mathbf{u}D_\mathbf{x}\widetilde{f}]$, since there are no $\mathbf{x}$-terms in $\boldsymbol{\lambda}\mathbf{u}$.

2. Since $\mathbf{x}\mapsto\widetilde{f}\begin{pmatrix}\mathbf{x}\\0\end{pmatrix}$ can be written $\mathbf{x}\mapsto f\begin{pmatrix}\mathbf{x}\\ \gamma(\mathbf{x})\end{pmatrix}$, the signature of $f|_Z$ at the critical point $\mathbf{z}_0$ is the signature of the Hessian matrix $[D^2_\mathbf{x}\widetilde{f}]$. Since this Hessian matrix appears as the matrix A in Lemma A14.2, the proof of part 2 follows immediately from the lemma.

The function $\mathbf{x}\mapsto\widetilde{f}\begin{pmatrix}\mathbf{x}\\ \mathbf{0}\end{pmatrix}$ is the original function f of Theorem 3.7.13, written in the parametrization $\mathbf{x}\mapsto\gamma(\mathbf{x})$. By definition, the signature of $f|_Z$ at the (constrained) critical point $\mathbf{z}_0$ is the signature of the Hessian matrix of $f\circ\gamma$ at the (unconstrained) critical point $\mathbf{x}_0$ of $f\circ\gamma$.

Lemma A14.2. *Let* Q *be the quadratic form on* $\mathbb{R}^{n+2m}$ *associated to the symmetric* $(n+2m)\times(n+2m)$ *real matrix*

$$M \overset{\text{def}}{=} \begin{array}{c c} \begin{array}{ccc} n & m & m \end{array} & \\ \begin{bmatrix} A & B^\top & 0\\ B & C & -I_m\\ 0 & -I_m & 0\end{bmatrix} & \begin{array}{c} n\\ m\\ m\end{array}\end{array}$$

and let q_A *be the quadratic form on* $\mathbb{R}^n$ *associated to the* $n\times n$ *matrix* A. *If* q_A *has signature* (p,q), *then* Q *has signature* $(p+m, q+m)$.

Proof of Lemma A14.2. Let Q_t be the quadratic form associated to the symmetric matrix

$$M_t \overset{\text{def}}{=} \begin{array}{c c} \begin{array}{ccc} n & m & m \end{array} & \\ \begin{bmatrix} A & tB^\top & 0\\ tB & tC & -I_m\\ 0 & -I_m & 0\end{bmatrix} & \begin{array}{c} n\\ m\\ m\end{array}\end{array}$$

so $Q_1 = Q$, and Q_0 is associated to

$$M_0 \stackrel{\text{def}}{=} \begin{array}{c} \begin{array}{ccc} n & m & m \end{array} \\ \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & -I_m \\ 0 & -I_m & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ n \\ m \\ m \end{array}.$$

Let the signature of Q_t be (p_t, q_t). Exercise A14.2 asks you to show that the signature of Q_0 is $(p+m, q+m)$, so the rank of M_0 is $p+q+2m$.

Since in the third column and the third row of M_t the only nonzero entry is $-I_m$, by row and column operations we can turn all the matrices M_t into M_0. Since row and column operations don't change the rank, all the matrices M_t have the same rank $p_t + q_t = p + q + 2m$.

This doesn't quite show that $p_t = p + m$ and $q_t = q + m$. But for t in a sufficiently small neighborhood of 0 we must have $p_t \geq p_0$ and $q_t \geq q_0$. Indeed, if $V^+ \subset \mathbb{R}^{n+2m}$ is a maximal subspace on which Q_0 is positive definite, then Q_t is still positive definite on V^+ for sufficiently small t, so $p_t \geq p_0$, and similarly $q_t \geq q_0$.

From

$$p_t \geq p + m, \quad q_t \geq q + m \quad \text{and} \quad p_t + q_t = p + q + 2m$$

we easily see that $p_t = p+m$ and $q_t = q+m$, in particular $p_1 = p+m$ and $q_1 = q+m$, which is the desired result. □

PAGE 764 [added Jan. 6, 2022] Equation A18.7: the second line should read

$$-\overline{g}(\mathbf{x}) = \sup |f|\, \mathbf{1}_X(\mathbf{x}) \quad \text{otherwise.}$$

PAGE 765 [added Jan. 6, 2022] Inequalities A18.13: The left side of the first inequality should not have absolute values.

PAGE 787 [new] Three lines before equation A21.54: the statement that the f_k are still R-integrable is not true. Fortunately the fix is easy. We have replaced everything from "To avoid the minor" to the end of the page by:

Theorem A17.2 (Fubini's theorem for Riemann integrals) shows that

$$U(f_k^{\mathbf{y}}) \underset{L}{=} L(f_k^{\mathbf{y}}) \quad \text{and} \quad U\big((f_k)_{\mathbf{x}}\big) \underset{L}{=} L\big((f_k)_{\mathbf{x}}\big), \qquad A21.54$$

Theorem A17.2: Since

$$U(f_{\mathbf{x}}) \geq L(f_{\mathbf{x}})$$

and the integrals are equal, the upper and lower sums in equation 21.54 are "Lebesgue equal". See Corollary 4.4.11.

and that both are L-integrable, so as far as Lebesgue integrals are concerned we can write

$$U(f_k^{\mathbf{y}}) \underset{L}{=} L(f_k^{\mathbf{y}}) = \int_{\mathbb{R}^m} f_k^{\mathbf{y}} |d^m\mathbf{y}| \quad \text{and} \quad U\big((f_k)_{\mathbf{x}}\big) \underset{L}{=} L\big((f_k)_{\mathbf{x}}\big) = \int_{\mathbb{R}^n} \big((f_k)_{\mathbf{x}}\big) |d^n\mathbf{x}|.$$

PAGE 793 [added Jan. 6, 2022] Exercise A21.1, first displayed exercise: $x \in [0,1]$, not $x \in \mathbb{R}$.

Page 799 [new] We have rewritten Lemma A23.4 to include a compact subset $K \subset U$:

Lemma A23.4. *Let $S \subset \mathbb{R}^n$ be a sphere, let $U \subset \mathbb{R}^k$ be open with $K \subset U$ compact, and let $\mathbf{f}: U \to \mathbb{R}^{n-k}$ be of class C^1. Then there exists $g \in G$ arbitrarily close to the identity such that the graph $\Gamma(\mathbf{f})$ is transversal to $g(S)$ on $\mathbf{f}(K)$.*

Minor errata

Page 29 [added 8/7/21] In two places Tartaglia's first name should have an accent: Niccolò, not Niccolo.

Page 76 [added Oct. 26, 2020] First margin note: "When solving big systems of linear equations", not "when solving big systems of linear questions".

Page 89 [added 8/7/21] Line 3: "how to play", not "how the play".

Page 92 [added Jan. 6, 2022] Proposition 1.5.21: the first sentence should read "Let X be a subset of $\mathbb{R}^n$ and let $f: X \to \mathbb{R}^m$ be a function. "

Page 94 [added Jan. 6, 2022] Theorem 1.5.26, part 3: in the line immediately before equation 1.5.39, x_0 should be $\mathbf{x}_0$.

Page 140 [added Oct. 26, 2020] The three sums in equation 1.8.15 should be for i from 1 to m, not to n: $\sum_{i=1}^{m}$, not $\sum_{i=1}^{n}$.

Page 165 Exercise 2.1.1, part a: "using the format of equation 2.1.2", not "using the format of Exercise 1.2.2".

Page 175 The discussion following Proposition 2.3.1 should have been labeled as a proof.

Page 188 [added Sept. 17, 2020] We will change the subheading at the bottom of the page to "Proof of Proposition and Definition 2.4.11".

Page 196 [added Sept. 17, 2020] In the second paragraph after Definition 2.5.9, $\mathbf{0}$ should be $\vec{\mathbf{0}}$.

Page 221 Line immediately before Definition 2.7.3: λTV should be $\lambda T\mathbf{v}$.

Page 285 In Definition 3.1.2., the words "smooth k-dimensional manifold" should be in italics.

Page 289 Line 1: "no longer than" rather than "shorter". In the last margin note, change "the two angles above" to "two angles". The angles are described in the exercise.

PAGE 290 Caption to Figure 3.1.9: "they can't move", not "they can't moved".

PAGE 302 Exercise 3.1.5: Replace "the set of equation $X_c = x^2 + y^3 = c$ by "the set X_c of equation $x^2 + y^3 = c$".

PAGE 311 There should be a $\square$ at the bottom of the page to indicate the end of the proof.

PAGE 312 [added 8/7/21] The first margin note has been expanded: Proposition 3.2.11 is true for all nonnegative integers n, k, and l. Saying that $\mathbf{g} : U \to M$ is C^1 is saying that as a map to $\mathbb{R}^n$, $\mathbf{g}$ is C^1, and its image just happens to lie in M.

PAGE 314 Exercise 3.2.11, part d: $F^{-1}(0)$ should be $F^{-1}([0])$.

PAGE 314 [added Sept. 17, 2020] Section 3.3, line 4 of first paragraph: we will replace "the function and its derivative" by "the function and this linear approximation".

PAGE 319 At the end of the first line, $a_k z^k$ should be $a_k x^k$.

PAGE 332 [added 8/7/21] The hint for Exercise 3.4.4 requires too much work. It should be

Exercise 3.4.4: Develop both

$$h\Big(af(0) + bf(h) + cf(2h)\Big) \text{ and } \int_0^h f(t)\,dt$$

as Taylor polynomials of degree 3 in h, and equate coefficients.

PAGE 342 [added 8/7/21] Exercise 3.5.17: We are removing the parentheses in $\vec{\mathbf{x}}^\top Q(\vec{\mathbf{x}})$, since it is a matrix multiplication. We are changing "an ellipse, a hyperbola ... " to "an ellipse, a hyperbola, an ellipsoid, ... " since the locus described might be a surface.

PAGE 347 [added 8/7/21] We are changing the conditions on t in equations 3.6.16, 3.6.18, and 3.6.19 so they are all the same: $|t| > 0$ sufficiently small.

PAGE 347 [added 8/7/21] Equation 3.6.19, after the first equality: $r(t\vec{\mathbf{h}})$ should be $r(t\vec{\mathbf{k}})$.

PAGE 347 [added 8/7/21] We are deleting the last margin note; W is not defined.

PAGE 358 [added 8/7/21] Line 2: "at all critical points", not "at all critical point".

PAGE 359 [added 8/7/21] Last margin note: the period before "The" should be deleted.

PAGE 381 [added 8/7/21] Figure 3.9.3: In the caption, F should be g. The z towards the top of the figure should be deleted.

PAGE 383 [added 8/7/21] First line of second paragraph: The reference should be to Example 3.1.19, not Definition 3.1.19.

PAGE 394 [added 8/7/21] The margin note for equation 3.9.68 should be next to that equation. The margin note for equation 3.9.69 should be on the next page.

PAGE 404 A period is missing at the end of Definition 4.1.3.

PAGE 408 [added 8/7/21] Equations 4.1.25: in the second equation, we are changing "lowest" to "least".

PAGE 419 [added 8/7/21] Equation 4.2.6: $\mu(\mathbf{x})$, not $\mu(x)$

PAGE 419 [added 8/7/21] Equation 4.2.7: to be consistent with equation 4.2.6, d^n not d^k.

PAGE 422 [added 8/7/21] Formula 4.2.19: In the denominator of the exponent, .5, not $,5$.

PAGE 429 [added 8/7/21] Proof of Corollary 4.3.10, lines 5-6: "of the first kind", not "of first kind". To make it clear that "for N sufficiently large"' applies to all cubes, we will rewrite the last sentence as "For N sufficiently large, cubes of the first kind have total volume $< \epsilon$ by Theorem 4.3.1, and cubes of the second kind have total volume $< \epsilon$ by Proposition 4.1.23."

PAGES 436-437 [added 8/7/21] In the last paragraph of the proof, we meant to write $\partial\mathcal{D}_N(\mathbb{R}^n)$, not $\delta\mathcal{D}_N(\mathbb{R}^n)$ (three occurrences).

PAGES 438 [added 8/7/21] We are changing Exercise 4.4.7 to match the solution: "Show that the set of numbers in $[0, 1]$ that in base 10 can be written using only the digits 1–6 has measure 0."

PAGE 460 [added 8/7/21] Limits in equation 4.7.3: the indicator functions are not necessary; the elements of $\mathcal{P}_N$ are subsets of X. But in equation 4.7.4 the indicator function on the right is required.

PAGE 465 [added 8/7/21] Equation 4.8.15: To be consistent with earlier notation, $A_{1,1}$ should be $A_{[1,1]}$.

PAGE 465 [added Jan. 6, 2022] Same for $A_{1,1}$ two lines before equation 4.8.15.

PAGE 468 [added Jan. 6, 2022] The first margin note should refer to Section 2.6, not 2.7.

PAGE 469 Proof of Theorem 4.8.9: To be consistent with earlier notation, $A_{1,1}$ should be $A_{[1,1]}$ (three occurrences).

PAGE 470 [added Oct. 26, 2020] Four lines after equation 4.8.39, two commas should be deleted: "The identity for $(1,2,3)$ is $(1)(2)(3)$", not "The identity for $(1,2,3)$ is $(1),(2),(3)$."

PAGES 471-472 [added 8/7/21] In one place on page 471 and four places on page 472, $\operatorname{Perm}(1,\dots,n)$ should be Perm_n (this notation is introduced at the top of page 470).

PAGE 476 [added 8/7/21] Equation 4.8.58: P^1AP should be $P^{-1}AP$.

PAGE 476 [added Sept. 17, 2020] Corollary 4.8.25 should begin "If A is an $n \times n$ matrix", not "Show that if A … ".

PAGE 528 [added 8/7/21] 2nd line from bottom: "unit circle".

PAGE 531 [added 8/7/21] In 5.2.4, the summation is from $i = 1$, not $n = 1$.

PAGE 536 [added 8/7/21] Second line of proof: $\mathbf{y}'_i$ not y'_i. Line before equation 5.2.27: "the set of unit vectors sequence of unit vectors" should be "the sequence of unit vectors". In the denominator of equation 5.2.27, the first $\mathbf{y}_i$ should be $\mathbf{y}'_i$; in the denominator of equation 5.2.28, the first $\mathbf{y}_j$ should be $\mathbf{y}'_j$.

PAGE 536 [added Oct. 26, 2020] After the proof of Lemma 5.2.12 we should have said that this concludes the proof of Theorem 5.2.11.

PAGE 536 [new] The left side of formula 5.2.24 is missing some parentheses; it should be $\gamma_1^{-1}B'_{r'_j}\big(\gamma_1(\mathbf{y}_j)\big)$.

PAGE 541 [added 8/7/21] In equation 5.3.21, the first two integrals should be $\int_{[0,A]}$ not $\int_0^A$.

PAGE 546 [added 8/7/21] Line before equation 5.3.48: "volume", not "area". Line after equation 5.3.48: by "above the ball" we meant "that projects to $B_R(\mathbf{0})$ (the ball of radius R centered at the origin) in $\mathbb{R}^3$".

PAGE 551 [added 8/7/21] Starting on this page, and on every subsequent odd page, the running head at the top of the page is wrong.

PAGE 553 [added Oct. 26, 2020] Left side of equation 5.4.15: the δ_r should have a tilde: $\widetilde{\delta}_r$.

PAGE 555 [added Jan. 6, 2022] The second line of equation 5.4.27 is missing a $|d^2\mathbf{u}|$ at the end. Two lines after equation , "second line" should be "third line".

PAGE 561 [added Jan. 6, 2022] In equation 5.5.3, to be consistent with our previous notation, $|d\mathbf{x}^{\ln 4/\ln 3}|$ should be $|d^{\ln 4/\ln 3}\mathbf{x}|$.

PAGE 566 [added Oct. 26, 2020] Example 6.1.3, line 2: "other words", not "other word".

PAGE 571 [added Oct. 26, 2020] About one-third down the page, in the sentence starting "Now consider": "for a k-form", not "for an k-form".

PAGE 571 [added 8/7/21] Equation 6.1.23: in the second line, the second sum should be $\sum_{i_2=1}^{n}$ not $\sum_{i_2}^{n}$.

PAGE 571 [added Oct. 26, 2020] Equation 6.1.23, third line: a close parenthesis is needed, for $\varphi(\vec{\mathbf{e}}_{i_1}, \dots, \vec{\mathbf{e}}_{i_k})$.

PAGE 585 [added 8/7/21] Line 6: "continuous function on", not "continuous function of".

PAGE 590 [added Oct. 26, 2020] There are two end-of-example symbols Δ. The material immediately after Example 6.4.3 should have been labeled as an example. We won't make that change now as it would confuse the numbering in this section.

PAGE 591 [added 8/7/21] Pages 591, lines 7–8: "change of basis matrix", not "change of matrix basis".

PAGE 593 [new] Two lines before equation 6.4.17: it should be "the columns of $[\mathbf{D}\gamma_1(\mathbf{u}_1)]$ and $[\mathbf{D}\gamma_2(\mathbf{u}_2)]$ are two bases".

PAGE 596 [added 8/7/21] Arrows are missing two lines before equation 6.4.29: "corresponds to $\det[\mathbf{D}\Phi(\mathbf{u}_1)] = \det[D_1\Phi(\mathbf{u}_1), \dots, D_k\Phi(\mathbf{u}_1)]$" should be "corresponds to $\det[\mathbf{D}\Phi(\mathbf{u}_1)] = \det[\overrightarrow{D_1\Phi}(\mathbf{u}_1), \dots, \overrightarrow{D_k\Phi}(\mathbf{u}_1)]$."

PAGE 606 [added 8/7/21] First margin note: the reference on the last line should be to Example 5.3.9, not 5.2.9.

PAGE 613 [added 8/7/21] Proposition 6.6.4: k-dimensional, not k-domensional.

PAGE 616 [added Oct. 26, 2020] Last line: "coming from" not "coming form".

PAGE 616 [added 8/7/21] First line: contrast, not constrast. Line 2: "locus Y consisting of ... ", not "locus consisting of ... "

PAGE 616 [added 8/7/21] Example 6.6.15, first paragraph: we need to specify that $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ are linearly independent.

PAGE 619 [added 8/7/21] In the paragraph beginning "Why do the inequalities ", we failed to define $\mathbf{f}$. After the first sentence we are adding: "Let $\mathbf{f} = \mathbf{0}$ define the manifold M".

PAGE 620 [added Oct. 26, 2020] Equation 6.6.17, left side, Ω should have a subscript $\mathbf{x}$: $\Omega^{\partial}_{\mathbf{x}}(\vec{\mathbf{v}})$, not $\Omega^{\partial}(\vec{\mathbf{v}})$

PAGE 621 [added Oct. 26, 2020] Third line of Example 6.6.25: it would be better to write "orientation by sgn det".

PAGE 622 [added Oct. 26, 2020] Example 6.6.27 should end with an end-of-example symbol $\triangle$.

PAGE 625 [added 8/7/21] Exercise 6.6.5: The notation ∂X should be $\partial_P X$ (four occurrences).

PAGE 629 [added 8/7/21] Caption for Figure 6.7.2, first line: Replace "The vectors" by "The unit vectors $\frac{\vec{\mathbf{x}}}{|\vec{\mathbf{x}}|}$ in the direction of $\vec{\mathbf{x}}$ ".

PAGE 639 [new] Exercise 6.8.6: evaluate on the 1-dimensional parallelogram $P_{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$, not on a vector.

PAGE 645 [added Oct. 26, 2020] Exercise 6.9.3, part c: parentheses are missing in the second displayed equation. It should be

$$(f^*\Phi_\xi)(\mathbf{x}) = \Phi_{\text{adj}([\mathbf{D}f(\mathbf{x})])\xi(\mathbf{x})}$$

PAGE 645 [added 8/7/21] Exercise 6.9.3: We are changing f to $\mathbf{f}$ since the function is vector valued.

PAGE 651 [added 8/7/21] Lines 1–2 of "Part 1": eliminate space in "diffeomorphism".

PAGE 653 [added 8/7/21] In equation 6.10.32, $e^{-1/(R^2-\mathbf{x}|^2)}$ should be $e^{-1/(R^2-|\mathbf{x}|^2)}$ (i.e., $|\mathbf{x}|^2$, not $\mathbf{x}|^2$).

PAGE 653 [new], one line after equation 6.10.33, "the ball V_i" should be "the ball U_i."

PAGE 656 [added 8/7/21] Line after inequality 6.10.53: Proposition 6.10.10, not Proposition A6.10.10.

PAGE 657 [new] Inequality 6.10.61 is missing two end parentheses.

PAGE 660 [added 8/7/21] Exercise 6.10.1: the word "compact" is unnecessary; a piece-with-boundary is compact by definition.

PAGE 670 [new] Last paragraph, line 3: "the stove", not "the the stove".

PAGE 676 [new] Equation 6.12.27: in the last line, the end parenthesis should be moved:

$$= -\operatorname{div}_{\mathbf{x}} \left(\frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \rho(\mathbf{x} + \vec{\mathbf{u}}) \vec{F}_3(\mathbf{u}) |d^3\mathbf{u}| \right)$$

PAGE 687 [new] Exercise 6.12.6: the first open parenthesis in the displayed equation should be deleted.

PAGE 691 [new] One line before Proposition 6.13.8: proposition, not lemma.

PAGE 700 [new] Exercise 6.12, part a: Sinces the input of a form is a parallelogram, the displayed equation should read

$$W_{\vec{F}}\Big(P_{\mathbf{x}}(\vec{\mathbf{v}})\Big) = \Phi_{\vec{F}}\left(P_{\mathbf{x}}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{\mathbf{v}}\right)\right).$$

PAGE 748 [added Jan. 6, 2022] Three lines from the bottom of the main text: a better reference might be Proposition 3.6.7.

PAGE 762 [added Jan. 6, 2022] Inequality A17.6: the $=$ on the third line should be $\geq$.

PAGE 769 [new] The left side of inequality A19.13 is missing an end parenthesis : $([\mathbf{D}\Phi(\mathbf{0})]$ should be $([\mathbf{D}\Phi(\mathbf{0})])$.

PAGE 784 [added Jan. 6, 2022] In the margin note, the first displayed equation is missing an integral sign; it should be $\sum_{l=1}^{\infty} \int_{\mathbb{R}^n} |h_l(\mathbf{x})||d^n\mathbf{x}|$.

PAGE 799 [new] Line immediately before the displayed expression A23.5: "given by" should be "by".

PAGE 802 [new] In Theorem A23.6 M should be $M \subset \mathbb{R}^n$.

Notes and clarifications

PAGE 85 Since "neighborhood" is not defined until page 86, we are changing the end of the margin note to read "every open interval centered at a rational number contains irrational numbers, and every open interval centered at an irrational number contains rational numbers."

PAGES 94–96 [added 8/7/21] In a number of places we fail to use arrows to distinguish points from vectors.

Page 110 Theorem 1.6.11 is true more generally for functions with values in $\mathbb{R}^m$ for any m, and in fact for functions with values in an arbitrary *metric space*. The same proof would work.

Page 157 [added Jan. 6, 2022] In Exercise 1.36, the definition of A applies to all three parts.

Page 275 Exercise 2.10.1: In keeping with our usual practice, all the F should be $\mathbf{F}$. But other authors may use different conventions.

Page 277 Exercise 2.10.15: Same comment as for Exercise 2.10.1.

Page 279 Exercise 2.15 repeats part c of Exercise 2.9.

Page 282 Exercise 2.37, part b: This should be stated as "if and only if": Let p_1 and p_2 be polynomials of degree k_1 and k_2. Then p_1 and p_2 are relatively prime if and only if there exist unique polynomials q_1 and q_2 of degree at most $k_2 - 1$ and $k_1 - 1$ such that $p_1q_1 + p_2q_2 = 1$.

Page 296 Proof of Theorem 3.1.16: That W is open follows from V being open and $\mathbf{f}$ continuous. Since $\mathbf{f}$ is continuous, if $\mathbf{y}$ is in W and $\mathbf{z}$ is a point in $\mathbb{R}^n$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\mathbf{z} - \mathbf{y}| < \delta \implies |\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{z})| < \epsilon$$

Since V is open, there exists $\epsilon > 0$ such that $B_\epsilon(\mathbf{f}(\mathbf{y}))$ is in V. Thus the ball of radius δ centered at $\mathbf{y}$ is a subset of W. So for every $\mathbf{y} \in W$, there exists δ such that $B_\delta(\mathbf{y})$ is a subset of W, so W is open.

Note for page 296: If V were not open, it would still be true that $W = \mathbf{f}^{-1}(V)$ would be open, but this requires a characterization of continuity not given in the book. To include it would require a definition of what it means for a subset V of a subset $X \subset \mathbb{R}^n$ to be open; our definitions of open and closed sets are limited to subsets of $\mathbb{R}^n$.

Page 304 Exercise 3.1.17: add "with the points restricted to a plane" ("In Example 3.1.8, with the points restricted to a plane, show that...").

Page 309 Proof of Proposition 3.2.7: U_1 is open since γ is a continuous function defined on an open set; see the note for the proof of Theorem 3.1.16.

Page 311 [new] We have changed Proposition and Definition 3.2.9 to include a definition of what it means for a map defined on a manifold to have locally Lipschitz derivative:

Proposition and Definition 3.2.9 (C^p map defined on manifold). *Let $M \subset \mathbb{R}^n$ be an m-dimensional manifold. A map $\mathbf{f} : M \to \mathbb{R}^k$ is of class C^p if every $\mathbf{x} \in M$ has a neighborhood $U \subset \mathbb{R}^n$ such that there exists a map $\widetilde{\mathbf{f}} : U \to \mathbb{R}^k$ of class C^p with $\widetilde{\mathbf{f}}|_{U\cap M} = \mathbf{f}|_{U\cap M}$.*

If $p \geq 1$, then $\mathbf{f}$ has locally Lipschitz derivative if U and $\widetilde{\mathbf{f}}$ can be chosen so that the derivative of $\widetilde{\mathbf{f}}$ is Lipschitz. Moreover,

$$[\mathbf{Df}(\mathbf{x})] : T_\mathbf{x}M \to \mathbb{R}^k \stackrel{\text{def}}{=} [\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]|_{T_\mathbf{x}M} \qquad 3.2.27$$

does not depend on the choice of the extension $\widetilde{\mathbf{f}}$ of $\mathbf{f}$.

PAGE 313 Exercise 3.2.7: To keep notation consistent with that used in the solution, replace $\mathbf{x}_0$ by $\mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

PAGE 314 Second sentence of Section 3.3: we are changing "at a point $\mathbf{x}$, the function and its derivative differ only ... " by "at a point $\mathbf{x}$, the function and this linear approximation differ only ... ".

PAGE 317 Last line of last margin note: Corollary 3.3.10 would be a better reference.

PAGE 330 [added 8/7/21] We have elaborated on the note for equation 3.4.22: Recall that $o(h^2)$ means *smaller* than h^2, so for the equation to be satisfied, the quadratic terms must vanish (the formula says nothing about higher-degree terms). It is handy to know that

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3.$$

PAGE 374 [added Jan. 6, 2022] Line before subheading "The covariance matrix": The correlation $\operatorname{corr}(f_R, g)$ can also be derived geometrically: f_R and f_B are orthogonal and of the same length ($\sigma(f_R) = \sigma(f_B)$), so the angle between their sum $g = f_R + f_B$ and either f_R or f_B is 45 degrees.

PAGE 387 [added 8/7/21] We are adding the following remark after Proposition 3.9.11:

Remark. The proof of Proposition 3.9.11 uses Proposition 3.9.10, where we require $c \neq 0$, but if $c = 0$ then equations 3.9.37 and 3.9.38 just become equations 3.9.30 and 3.9.31. $\triangle$

PAGE 427 [added 8/7/21] Proof of Corollary 4.3.8, first line: We are using the statement that the image of a compact set by a continuous map is compact, which we neglected to state and prove in Chapter 1. This can be proved using Heine-Borel and open covers (for instance, Theorem 1.11.15 in *Functional Analysis volume 1: A Gentle Introduction* by Dzung Ha.) It can also be proved as follows, using the Bolzano-Weierstrass theorem:

Let $f : X \to Y$ be a continuous map, with X compact. Let $n \mapsto y_n$ be a sequence in $f(X)$, so that there exists x_n such that $f(x_n) = y_n$. Since X is compact, there exists a convergent subsequence $i \mapsto x_{n_i}$ converging to $x_\infty \in X$. Since f is continuous, $i \mapsto y_{n_i}$ converges to $f(x_\infty)$, so Y is compact.

PAGES 496–497 [added Jan. 6, 2022] Line 3 page 496 and Exercise 4.10.6 page 497: we should have also asked you to show that the derivative is Lipschitz in $\overset{\circ}{X}$.

Page 504 [added Jan. 6, 2022] Siqi Wang suggested that the proof of Theorem 4.11.7 can be simplified. In any future edition we will replace the part starting on page 504 by the following:

By inequality 4.11.25, the integral of $|H_l - H_M|$ is less than ϵ, so we only need to consider the integral of $|[H_l]_R - H_M|$. Outside $B_R(\mathbf{0})$ we have $H_M = 0$ and $[H_l]_R = 0$, so we need only consider

$$\int_{B_R(\mathbf{0})} \big|[H_l]_R - H_M\big|(\mathbf{x})\,|d^n\mathbf{x}|.$$

For any $\mathbf{x} \in B_R(\mathbf{0})$, either

- $|H_l(\mathbf{x})| \le R$, and then $[H_l]_R(\mathbf{x}) = H_l(\mathbf{x})$, or
- $|H_l(\mathbf{x})| > R$, and then either $\begin{cases} H_M(\mathbf{x}) < R < H_l(\mathbf{x}) & \text{if } H_l(\mathbf{x}) > 0 \\ H_M(\mathbf{x}) > -R > H_l(\mathbf{x}) & \text{if } H_l(\mathbf{x}) < 0. \end{cases}$

In either case, $\big|[H_l]_R - H_M\big|(\mathbf{x}) \le \big|H_l(\mathbf{x}) - H_M(\mathbf{x})\big|$, so

$$\begin{aligned}\int_{B_R(\mathbf{0})} \big|[H_l]_R - H_M\big|(\mathbf{x})|d^n\mathbf{x}| &\le \int_{B_R(\mathbf{0})} \big|H_l(\mathbf{x}) - H_M(\mathbf{x})\big||d^n\mathbf{x}| \\ &\le \int_{\mathbb{R}^n} \big|H_l(\mathbf{x}) - H_M(\mathbf{x})\big||d^n\mathbf{x}| < \epsilon.\end{aligned}$$

So

$$\lim_{l\to\infty} \int_{\mathbb{R}^n} (H_l(\mathbf{x}) - [H_l]_R(\mathbf{x}))\,|d^n\mathbf{x}| = 0.$$

Since (by equation 4.11.27)

$$\lim_{l\to\infty} \int_{\mathbb{R}^n} [H_l]_R(\mathbf{x}))\,|d^n\mathbf{x}| = 0, \quad \text{this gives} \quad \lim_{l\to\infty} \int_{\mathbb{R}^n} H_l(\mathbf{x})\,|d^n\mathbf{x}| = 0.$$

This completes the proof of Theorem 4.11.7. □

Page 507 [added Jan. 6, 2022] In the proof of Proposition 4.11.14, we failed to specify why inequality 4.11.19 is satisfied:

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |af + bg|(\mathbf{x})|d^n\mathbf{x}| \le |a| \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\mathbf{x})|d^n\mathbf{x}| + |b| \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_g(\mathbf{x})|d^n\mathbf{x}| < \infty.$$

Page 514 [added 8/7/21] We are adding a new margin note for Example 4.11.25:

The Gaussian with its standard normalization is the normal distribution

$$\mu(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2};$$

see equation 4.2.14. The function e^{-ax^2} is a variant of the Gaussian.

Page 514 [added 8/7/21] We are adding a new margin note:

Equation 4.11.89: for the integral defining $\widehat{M^p f}$ to converge, f must decrease at infinity sufficiently rapidly that it is still L-integrable after multiplication by $(2\pi i x)^p$.

PAGE 515 [added 8/7/21] We are adding a new margin note at the top of the page: Integration by parts:

$$\int_a^b f(x)G(x)\,dx = [F(x)G(x)]_a^b - \int_a^b F(x)g(x)\,dx,$$

where

$$\begin{aligned} &a = -\infty, \quad b = \infty, \\ &f(x) = 2xe^{-ax^2}, \quad F(x) = \frac{-e^{-ax^2}}{a} \\ &g(x) = 2\pi i\xi e^{2\pi i x\xi}, \quad G = e^{2\pi i x\xi}. \end{aligned}$$

PAGE 525 [new] The matrix $T^\top T$ is called the *Gram matrix*; $\det(T^\top T)$ is called the *Gram determinant*.

PAGE 555 [new] We should start Theorem 5.4.4 with "The Taylor polynomial of degree 1 of the function $t \mapsto \text{Area}\, S_t$ is". We have rewritten the proof:

Proof. Let $U \subset \mathbb{R}^2$ be open and let $\gamma : U \to S$ be a parametrization. Set $\vec{\mathbf{w}}(\mathbf{u}) \stackrel{\text{def}}{=} \vec{W}(\gamma(\mathbf{u}))$. Then $\gamma_t : \mathbf{u} \mapsto \gamma(\mathbf{u}) + t\vec{\mathbf{w}}(\mathbf{u})$ parametrizes S_t and

$$\begin{aligned} \text{Area}\, S_t &\underbrace{=}_{\text{Def. 5.3.2}} \int_U \sqrt{\det\Big([\mathbf{D}(\gamma_t)(\mathbf{u})]^\top [\mathbf{D}(\gamma_t)(\mathbf{u})]\Big)}\,|d^2\mathbf{u}| \\ &= \int_U \sqrt{\det\Big(\big([\mathbf{D}\gamma(\mathbf{u})]^\top + t[\mathbf{D}\vec{\mathbf{w}}(\mathbf{u})]^\top\big)\big([\mathbf{D}\gamma(\mathbf{u})] + t[\mathbf{D}\vec{\mathbf{w}}(\mathbf{u})]\big)\Big)}|d^2\mathbf{u}| \qquad 5.4.27 \\ &= \int_U \sqrt{\det\Big(\underbrace{[\mathbf{D}\gamma(\mathbf{u})]^\top[\mathbf{D}\gamma(\mathbf{u})]}_{A(\mathbf{u})} + t\Big(\underbrace{[\mathbf{D}\vec{\mathbf{w}}(\mathbf{u})]^\top[\mathbf{D}\gamma(\mathbf{u})] + [\mathbf{D}\gamma(\mathbf{u})]^\top[\mathbf{D}\vec{\mathbf{w}}(\mathbf{u})]}_{B(\mathbf{u})}\Big)\Big)}\,|d^2\mathbf{u}| + o(t), \end{aligned}$$

For any two matrices A and B with A invertible, we have

Equation 5.4.28: Equality 1 is $f(\mathbf{x}+\mathbf{h}) = \mathbf{f}(\mathbf{x}) + [\mathbf{D}f(\mathbf{x})]\mathbf{h} + o(|\mathbf{h}|)$.

Equality 2 is Theorem 4.8.15.

Equality 3 is equation 3.4.9 with $x = t\,\text{tr}(A^{-1}B)$ and $m = 1/2$.

$$\begin{aligned} \sqrt{\det(A+tB)} &\underset{1}{=} \sqrt{\det A + t[\mathbf{D}\det(A)]B + o(t)} \\ &\underset{2}{=} \sqrt{\det A + t(\det A)\,\text{tr}(A^{-1}B)} + o(t) \\ &\underset{3}{=} \sqrt{\det A}\left(1 + \frac{t}{2}\,\text{tr}(A^{-1}B))\right) + o(t) \qquad 5.4.28 \\ &= \sqrt{\det A} + \frac{t}{2}\sqrt{\det A}\,\text{tr}(A^{-1}B)) + o(t). \end{aligned}$$

So we can rewrite equation 5.4.27 as

$$\begin{aligned}\operatorname{Area} S_t &= \int_U \left(\sqrt{\det A(\mathbf{u})} + \frac{t}{2}\sqrt{\det A(\mathbf{u})}\,\operatorname{tr}\Big(A(\mathbf{u})^{-1}B(\mathbf{u})\Big)\right)|d^2\mathbf{u}| + o(t) \\ &= \underbrace{\int_U \sqrt{\det A(\mathbf{u})}|d^2\mathbf{u}|}_{\operatorname{Area} S} + \underbrace{\frac{t}{2}\int_U \sqrt{\det A(\mathbf{u})}\,\operatorname{tr}\Big(A(\mathbf{u})^{-1}B(\mathbf{u})\Big)|d^2\mathbf{u}|}_{\text{degree 1 terms of } t\mapsto \operatorname{Area} S_t} + o(t).\end{aligned} \qquad 5.4.29$$

Therefore to prove the theorem we need to show that

$$-2t\int_S \vec{H}(\mathbf{x})\cdot\vec{W}(\mathbf{x})\,|d^2\mathbf{x}| = \frac{t}{2}\int_U \sqrt{\det A(\mathbf{u})}\,\operatorname{tr}\Big(A(\mathbf{u})^{-1}B(\mathbf{u})\Big)\,|d^2\mathbf{u}| \qquad 5.4.30$$

In equation 5.4.31

$$\vec{H}(\gamma(\mathbf{u}))\cdot\vec{W}(\mathbf{u})$$

plays the role of $f(\gamma(\mathbf{u}))$ in equation 5.3.5.

By equation 5.3.5,

$$\int_S \vec{H}(\mathbf{x})\cdot\vec{W}\mathbf{x})\,|d^2\mathbf{x}| = \int_U \vec{H}(\gamma(\mathbf{u}))\cdot\vec{\mathbf{w}}(\mathbf{u})\sqrt{\det A(\mathbf{u})}\,|d^2\mathbf{u}|. \qquad 5.4.31$$

So the theorem will be proved if we can show that

$$-4\int_U \sqrt{\det A(\mathbf{u})}\vec{H}(\gamma(\mathbf{u}))\cdot\vec{\mathbf{w}}(\mathbf{u})|d^2\mathbf{u}| = \int_U \sqrt{\det A(\mathbf{u})}\,\operatorname{tr}\Big(A(\mathbf{u})^{-1}B(\mathbf{u})\Big)|d^2\mathbf{u}|.$$

We will show that the integrands are equal, i.e.,

$$-4\Big(\vec{H}(\gamma(\mathbf{u}))\Big)\cdot\vec{\mathbf{w}}(\mathbf{u}) = \operatorname{tr}\Big(A(\mathbf{u})^{-1}B(\mathbf{u})\Big) \qquad 5.4.32$$

Equation 5.4.32: this is a pointwise statement; it is enough to prove it for any $\mathbf{x} = \gamma(\mathbf{u})$ in S. We will prove it for $\gamma(\mathbf{0})$.

Since integrals are independent of parametrization (Proposition 5.3.3) we can choose the parametrization

This parametrization is equivalent to rotating and translating S to move $\mathbf{x}$ to the origin, and moving $T_\mathbf{x}S$ to the (x,y)-plane; see equation 5.4.4.

$$\gamma\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\\f\begin{pmatrix}x\\y\end{pmatrix}\end{pmatrix}, \quad \text{where } f\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{2}(ax^2+by^2)+o(x^2+y^2)$$

is of class C^2. Compute the right side of equation 5.4.32 at $\mathbf{0}$. We have

$$A(\mathbf{0}) = \left[\mathbf{D}\gamma\begin{pmatrix}0\\0\end{pmatrix}\right]^\top\left[\mathbf{D}\gamma\begin{pmatrix}0\\0\end{pmatrix}\right] = \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}. \qquad 5.4.33$$

To compute $B(\mathbf{0})$ we first compute

$$B(\mathbf{u}) = [\mathbf{D}\vec{\mathbf{w}}(\mathbf{u})]^\top[\mathbf{D}\gamma(\mathbf{u})] + [\mathbf{D}\gamma(\mathbf{u})]^\top[\mathbf{D}\vec{\mathbf{w}}(\mathbf{u})].$$

Since $\vec{W}$ is orthogonal to S, it can be written as a multiple of the cross product of the partial derivatives of γ by some scalar-valued function α:

$$\vec{\mathbf{w}}\begin{pmatrix}x\\y\end{pmatrix} = \vec{W}\left(\gamma\begin{pmatrix}x\\y\end{pmatrix}\right) = \alpha\begin{pmatrix}x\\y\end{pmatrix}\left(\begin{bmatrix}1\\0\\ax+o(|x|+|y|)\end{bmatrix}\times\begin{bmatrix}0\\1\\by+o(|x|+|y|)\end{bmatrix}\right) = \alpha\begin{pmatrix}x\\y\end{pmatrix}\begin{bmatrix}-ax+o(|x|+|y|)\\-by+o(|x|+|y|)\\1\end{bmatrix}.$$

Ignoring higher-degree terms, we get

$$\left[\mathbf{D}\vec{\mathbf{w}}\begin{pmatrix}0\\0\end{pmatrix}\right] = \begin{bmatrix} -a\alpha\begin{pmatrix}0\\0\end{pmatrix} & 0 \\ 0 & -b\alpha\begin{pmatrix}0\\0\end{pmatrix} \\ D_1\alpha\begin{pmatrix}0\\0\end{pmatrix} & D_2\alpha\begin{pmatrix}0\\0\end{pmatrix}\end{bmatrix}, \text{ so } \left[\mathbf{D}\vec{\mathbf{w}}\begin{pmatrix}0\\0\end{pmatrix}\right]^\top \left[\mathbf{D}\gamma\begin{pmatrix}0\\0\end{pmatrix}\right] = \begin{bmatrix} -a\alpha\begin{pmatrix}0\\0\end{pmatrix} & 0 \\ 0 & -b\alpha\begin{pmatrix}0\\0\end{pmatrix}\end{bmatrix}.$$

Since $\mathrm{A}(0)^{-1}$ is the identity and the two terms of $B(\mathbf{0})$ are equal, at $(\mathbf{0})$ the right side of equation 5.4.32 is $\operatorname{tr}\Big(A(\mathbf{0})^{-1}B(\mathbf{0})\Big) = -2\alpha\begin{pmatrix}0\\0\end{pmatrix}(a+b)$.

By equation 3.9.30 the left side of 5.4.32 is the same:

$$-4\Big(\vec{H}(\gamma(\mathbf{u}))\Big)\cdot\vec{\mathbf{w}}(\mathbf{u}) = -4\begin{bmatrix}0\\0\\\frac{1}{2}(a+b)\end{bmatrix}\cdot\begin{bmatrix}0\\0\\\alpha\begin{pmatrix}0\\0\end{pmatrix}\end{bmatrix} = -2\alpha\begin{pmatrix}0\\0\end{pmatrix}(a+b). \quad \square$$

PAGE 584 [new] To be consistent with the notation of equation 6.3.3, we are changing $\Omega^{\vec{\mathbf{t}}}_{\mathbf{x}}(\vec{\mathbf{v}})$ in equation 6.3.4 to $\Omega^{\vec{\mathbf{t}}}_{\mathbf{x}}(\mathbf{x};\vec{\mathbf{v}})$ and $\Omega^{\vec{\mathbf{n}}}_{\mathbf{x}}(\vec{\mathbf{v}}_1,\vec{\mathbf{v}}_2)$ in equation 6.3.5 to $\Omega^{\vec{\mathbf{n}}}_{\mathbf{x}}(\mathbf{x};\vec{\mathbf{v}}_1,\vec{\mathbf{v}}_2)$. In part 3 of Proposition 6.3.4, we are specifying what it means for $\vec{\mathbf{t}}$ to be a continuous tangent nonvanishing vector field: "Let $\vec{\mathbf{t}}$ be a nonvanishing tangent vector field on C, i.e., a continuous map $\vec{\mathbf{t}}: C \to \mathbb{R}^n$ with $\vec{\mathbf{t}}(\mathbf{x}) \in T_{\mathbf{x}}C$ and $\vec{\mathbf{t}}(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in C$."

PAGE 620 [added 8/7/21] Figure 6.6.10: We are replacing the last sentence of the caption by "Thus the oriented boundary of P consists of the point $\mathbf{b}$ with the orientation $+1$ and the point $\mathbf{a}$ with the orientation -1; we denote this $+\mathbf{b}-\mathbf{a}$. This notation will be generalized in Proposition 6.6.26."

PAGE 639 [added 8/7/21] Exercise 6.8.9, part b: this is wrong if "twice differentiable" is taken to mean that the first partials are differentiable. It is right if it is taken to mean that the first partials themselves have partial derivatives. We are replacing it by

b. Show that this is not necessarily true if we only assume that all second partials of f exist everywhere.

PAGES 652–653 [added 8/7/21] In a number of places we fail to use arrows to distinguish points from vectors.

PAGE 691 [new] Definition 6.13.7 makes sense in the degenerate case where the point $\mathbf{x}$ is in the span of the $\vec{\mathbf{v}}_i$ or the $\vec{\mathbf{v}}_i$ are themselves linearly dependent; the cone is then flat. But the proof of Proposition 6.3.18 only makes sense in the nondegenerate case, so that there is such a thing as an outward-pointing vector. We will say that in the degenerate case, the boundary of the cone as a parametrized domain is defined by equation 6.13.12.

PAGE 702 [new] Exercise 6.27: We should have specified cgs units; in SI there is an extra ϵ_0.

PAGE 777 [added Jan. 6, 2022] Equation A21.4: The

$$= \int_Q \inf(f_k(\mathbf{x}), K)||d^n\mathbf{x}| + \int_Q \sup(f_k(\mathbf{x}), K)||d^n\mathbf{x}| - K$$

is unnecessary (and we don't need the "since for any numbers a and b, we have $a = \inf(a,b) + \sup(a,b) - b$" before the equation).

PAGE 802 [new] We have modified the proof of Theorem A23.6 slightly:

Proof of Theorem A23.6. Exercise A23.3 asks you to show that it is enough to prove this result if $U \subset \mathbb{R}^p$ is open, $\mathbf{g}: U \to \mathbb{R}^{n-p}$ for some n is a C^1 mapping, and Y is a compact part of the graph $\Gamma(\mathbf{g})$. Exercise A23.4 asks you to show that there exists C_1 such that for any $r > 0$ (however small), Y can be covered by C_1/r^p balls of radius r.

Set

$$C_2 = \sup_Y |[\mathbf{D}f(\mathbf{x})]|, \qquad A23.9$$

Let $B \subset \mathbb{R}^n$ be a ball of radius r centered at $\mathbf{x} \in \Gamma(\mathbf{g})$. Then $f(B \cap Y)$ is contained in the ball centered at $f(\mathbf{x})$ of radius $C_2 r$. Thus we see that $f(Y)$ is covered by $C_1 r^{-p}$ balls of radius $C_2 r$. By Definition 5.2.1, $f(Y)$ has q-dimensional volume 0 if

$$\left(\frac{C_1}{r^p}\right)(C_2 r)^q \qquad A23.10$$

can be made arbitrarily small, which is the case if $q > p$: then the product tends to 0 as $r \to 0$. $\square$

PAGE 803 [new] We made two changes to the proof of Proposition 6.10.10. In the first paragraph below, we introduced a neighborhood with compact closure of the nonsmooth part of the boundary, so that we get transversality where it matters. In the third paragraph, adding "and to their intersection" is needed to guarantee that the trimmed piece will be a piece-with-corners.

Proof of Proposition 6.10.10. Using Heine-Borel, cover $\partial^{ns}X$ by finitely many balls $B_i \stackrel{\text{def}}{=} B_{r_i}(\mathbf{x}_i),\ i = 1,\dots,p$. Choose a neighborhood W of $\partial^{ns}X$ with closure $\overline{W}$ contained in $\cup_i B_i$. Then $\partial^s X \cap (X - W)$ is compact. Note that if we move these balls sufficiently little and modify their radii sufficiently little, the new balls will still cover $\overline{W}$, in particular $\partial^{ns}(X)$.

By Theorem A23.3, we can move $\mathbf{x}_1$ to $\mathbf{x}_1'$ by an arbitrarily small rigid motion, and modify r_1 arbitrarily little, so that if we write $B_1' = B_{r_1'}(\mathbf{x}_1')$, then $\partial B_i'$ is transversal to $\partial^s X$ on $\partial^s(X) \cap (X - W)$, and $B_1', B_2, \dots, B_p$ still cover $\overline{W}$.

Now let B_2' be a similar small modification of B_2, so that $\partial B_2'$ is transversal to $\partial B_1'$, to $\partial^s X$ on $\partial^s(X) \cap (X - W)$, and to their intersection, and so that $B_1', B_2', B_3, \dots, B_p$ still cover $\overline{W}$.

Continue this way, making each $\partial B_i'$ transversal to all the previous $\partial B_j'$ and to their finite intersections, as well as to $\partial^s X$ on $\partial^s(X) \cap (X - W)$. When you get to B_p, you will be done: the trimmed piece $X - \cup_i B_i'$ will be a piece-with-corners. $\square$

PAGE 803 [new] Exercises A23.4: the hint should say "See the proof of Proposition 5.2.2."