Appendix D3
Fundamental groups, amalgamated sums, and HNN extensions

The Klein-Maskit combination theorems discussed in Section 11.11 are a vital tool in Thurston's proof of the hyperbolization of Haken manifolds. The first combination theorem requires *amalgamated sums*, which most readers will have encountered when proving van Kampen's theorem. The second requires the analogous *HNN extensions* that arise when gluing a space to itself rather than to a different space. That material is not harder but it is less standard. This appendix discusses both topics.

We begin in Section D3.1 with a review of covering space theory, mainly to set the notation we will need, and also to avoid traps associated with ignoring base points. But we have two other motivations. One is that the usual proof of van Kampen's theorem both more involved and less general than it needs to be, because it is based on loops and paths rather than the universal properties of universal covering spaces.

The other is that I wish to emphasize the similarities of Galois theory and covering space theory. The choice of a universal covering space for a space X corresponds to the choice of a separable algebraic closure $\overline{k}$ for a field k; the absolute Galois group $\mathrm{Aut}_k(\overline{k})$ corresponds to the fundamental group. In Galois theory, $\mathrm{Aut}_k(\overline{k})$ is a profinite group, and the category of étale algebras over k (i.e., finite direct sums of finite separable field extensions) is equivalent to the category of finite sets on which $\mathrm{Aut}_k(\overline{k})$ operates continuously. See [23] for a further development of these ideas, which are the foundation of the large but difficult literature on the étale fundamental group. The étale topology was invented by Grothendieck; this appendix is influenced by his ideas.

D3.1 Covering space theory: a crash course

A *covering map* $f : Y \to X$ is a locally trivial map with discrete fibers. This means that

1. The fiber $f^{-1}(x)$ is discrete for every $x \in X$.
2. Every $x \in X$ has a neighborhood U such that there exists a homeomorphism $\varphi : f^{-1}(U) \to f^{-1}(x) \times U$ satisfying $f(y) = pr_2(\varphi(y))$ for all $y \in f^{-1}(U)$, and $\varphi(y) = (y, x)$ for $y \in f^{-1}(x)$, where $pr_2 : f^{-1}(x) \times U \to U$ denotes projection to the second factor U.

If $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are two covering maps, we denote by $C_X(Y_1, Y_2)$ the space of continuous maps $g : Y_1 \to Y_2$ such that $f_2 \circ g = f_1$.

Exercise D3.1.1

1. Show that elements of $C_X(Y_1, Y_2)$ are themselves covering maps.
2. Suppose that Y_1 is connected, and let $y \in Y_1$ be a point. Show that if $g_1, g_2 \in C_X(Y_1, Y_2)$ satisfy $g_1(y) = g_2(y)$, then $g_1 = g_2$. $\diamondsuit$

If $Y_1 = Y_2 = Y$ the group of $g \in C_X(Y, Y)$ that are homeomorphisms is called the *group of deck transformations*, denoted $\mathrm{Aut}_X(Y)$. You should think of it as an analogue of the Galois group of a field extension.

A covering map $f : Y \to X$ is *trivial* if the neighborhood U of $x \in X$ can be taken to be X, i.e., if Y is X times a discrete set. Let X be a connected space. A *universal covering space* of X is a connected space $\widetilde{X}$ together with a covering map $p : \widetilde{X} \to X$ such that for any covering map $f : Y \to X$ with Y connected, there exists a covering map $g : \widetilde{X} \to Y$ satisfying $p = f \circ g$; this map p is the *universal covering map*.

There is a difficulty right at the beginning: spaces that are locally complicated, like the Hawaiian earring depicted in Figure A7.2.1 of Volume 1, don't have universal covering spaces. We will not be applying the theorems of this appendix to such spaces, but they are relevant to this book: limit sets of Kleinian groups and Julia sets of rational functions are usually just as pathological as the Hawaiian earring and often worse.

We will work in the category of *UC spaces*: spaces that have universal covering spaces. All the spaces in this appendix will be assumed to be Hausdorff (or proved to be Hausdorff if necessary).

The standard construction of the universal covering space

The standard construction of the universal covering space $\widetilde{X}$ requires that X be CCC: *connected, locally connected, and semi-locally simply connected*. However, there are UC spaces that do not have these properties (see Example D3.1.7).

It is wrong to speak of *the* covering space of a topological space unless one chooses a base point: there is no functor associating $\widetilde{X}$ to X.

Example D3.1.2 (Universal covering map not a functor) Consider the Hopf fibration $h\colon S^3 \to S^2$, with fibers homeomorphic to circles. If there were a functorial universal covering map, there would be a "universal covering space fiber by fiber" $\widetilde{h}\colon X \to S^2$ with some appropriate X and with fibers homeomorphic to $\mathbb{R}$. Since the fibers would be contractible, such a fibration would have a section $\sigma\colon S^2 \to X$ of $\widetilde{h}$, which would project