Vector Calculus, Linear Algebra and Differential Forms: A Unified Approach 5th Edition

Complete List of Errata and Notes as of May 22, 2018

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New items are marked in red. The list is divided into three sections: errata; minor typos; and notes and clarifications.

Errata

Page 73 [added Sept. 15, 2017] Proposition 1.4.11: "$\vec{\mathbf{a}}$ a vector in $\mathbb{R}^m$" should be "$\vec{\mathbf{b}}$ a vector in $\mathbb{R}^m$".

Page 79 The far right side of equation 1.4.52 should be $|\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})|$, not $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})|$; the volume is a number.

Page 107 [added Sept. 29, 2017] 4th paragraph, line 7: "saying that the x_m have a limit in $[0, 1/2)$" should be " ... have a limit in $[0, 1)$".

Page 110 Inequalities 1.6.14: The second $|\mathbf{x}_{i_j} - \mathbf{a}| < \delta$ should be $|\mathbf{y}_{i_j} - \mathbf{a}| < \delta$.

Page 113 Fifth paragraph, first line: "The disc $\{\, z \in \mathbb{C} \mid |z| \le R \,\}$", not "The disc $\{\, z \in \mathbb{C} \mid z \le R \,\}$"

Page 132 In the margin note that starts "We write equation 1.7.49" we should have referred to "lengths of matrices" rather than "absolute values".

Page 134 In the last equation of the proof (just before the exercises) $[0]$ should be 0 (the 0 is scalar, not a matrix).

Page 150 The proof of Theorem 1.9.8. is an application of Theorem 1.6.13, the mean value theorem in one variable, not an application of Theorem 1.9.1.

Page 170 Corollary 2.2.7 also uses Proposition 1.3.13; the proof of that proposition shows that if the linear transformation "multiplication by A"

is bijective, its inverse mapping is linear and so has a matrix; this matrix is a two-sided inverse of A.

Page 170 In the margin we define a *pivotal row* by saying say that when a row of $\widetilde{A}$ contains a pivotal 1, the corresponding row of A is a pivotal row. Because we use row operations when row reducing a matrix, "corresponding row" is not well defined. We will avoid the term in the future (although one could define pivotal row using the transpose).

Page 171 To avoid the use of "pivotal row" (see the erratum for page 170), we will replace formula 2.2.11 by

$$\begin{aligned} A \text{ one to one } &\iff \text{ every column of } A \text{ contains a pivotal 1} \\ &\iff \text{ every row of } A \text{ contains a pivotal 1} \\ &\iff A \text{ is onto.} \end{aligned} \tag{2.2.11}$$

and we will change the first margin note to " ... which has a pivotal 1 in each column but not in each row".

Page 187 There are several mistakes of sign in the third displayed equation of Example 2.4.17. It should be

$$\begin{aligned} \int_0^\pi \sin nx \sin mx \, dx &= \frac{1}{2}\int_0^\pi \Big((\cos(n-m)x - \cos(n+m)x) \Big)\, dx \\ &= \frac{1}{2}\left(\left[\frac{\sin(n-m)x}{n-m}\right]_0^\pi - \left[\frac{\sin(n+m)x}{n+m}\right]_0^\pi \right) = 0. \end{aligned}$$

Page 187 Next-to-last line, in the displayed equation,

$$+\dots a_k \sin n_k x \quad \text{should be} \quad + \dots + a_k \sin n_k x.$$

Page 194 Two lines before equations 2.5.6: "for $\vec{\mathbf{v}}_1$ ... the first entry is -1" should be "for $\vec{\mathbf{v}}_1$ ... the first entry is -2". " ... the corresponding entries for $\vec{\mathbf{v}}_2$ are -3, -2, and 0" should be " ... the corresponding entries for $\vec{\mathbf{v}}_2$ are are -1, 1, and 0."

Page 197 To avoid the use of "pivotal row" (see the erratum for page 170), we will replaced "the pivotal rows of A are linearly independent" by "the rows of $\widetilde{A}$ containing pivotal 1's are linearly independent".

Page 212 If and when the book is reprinted, we will move the discussion of dimension, including Proposition and Definition 2.6.21, immediately before the subsection "Matrix with respect to a basis and change of basis". In Proposition and Definition 2.6.17 on the change of basis matrix, V is said to be n-dimensional, but with the current order of discussion, dimension isn't yet defined, and we don't yet know that $\{\mathbf{v}\}$ and $\{\mathbf{v}'\}$ both have n elements. Fortunately, the proof of Proposition and Definition 2.6.21 requires only Proposition 2.6.15 and the fact that $\Phi^{-1}_{\{\mathbf{w}\}}$ and $\Phi^{-1}_{\{\mathbf{v}\}}$ are linear, which follows from equations 1.3.21 and 1.3.22.

PAGE 229 [added March 25, 2017] Three lines after equation 2.7.40, we speak of $g(A\vec{\mathbf{v}})$, but g is defined on Δ, the set of unit vectors in Q, and there is no reason to think that $A\vec{\mathbf{v}}$ is a unit vector. To deal with this, we are replacing the first three paragraphs of the proof of Theorem 2.7.10 by:

Let $Q \subset \mathbb{R}^n$ be the "quadrant" $\vec{\mathbf{w}} \geq \vec{\mathbf{0}}$, set $Q^* \stackrel{\text{def}}{=} Q - \{\vec{\mathbf{0}}\}$, and let Δ be the set of unit vectors in Q. If $\vec{\mathbf{w}} \in \Delta$, then $\vec{\mathbf{w}} \geq \vec{\mathbf{0}}$ and $\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$, so (by Lemma 2.7.11) $A\vec{\mathbf{w}} > \vec{\mathbf{0}}$.

Consider the function $g : Q^* \to \mathbb{R}$ given by

$$g : \vec{\mathbf{w}} \mapsto \inf \left\{ \frac{(A\vec{\mathbf{w}})_1}{w_1}, \frac{(A\vec{\mathbf{w}})_2}{w_2}, \dots, \frac{(A\vec{\mathbf{w}})_n}{w_n} \right\}; \qquad 2.7.39$$

then $g(\vec{\mathbf{w}})\vec{\mathbf{w}} \leq A\vec{\mathbf{w}}$ for all $\vec{\mathbf{w}} \in Q^*$, and $g(\vec{\mathbf{w}})$ is the largest number for which this is true. Note that $g(\vec{\mathbf{w}}) = g(\vec{\mathbf{w}}/|\vec{\mathbf{w}}|)$ for all $\vec{\mathbf{w}} \in Q^*$.

Since g is an infimum of finitely many continuous functions $Q^* \to \mathbb{R}$, the function g is continuous. Since Δ is compact, g achieves its maximum on Δ at some $\vec{\mathbf{v}} \in \Delta$, which also achieves the maximum of g on Q^*. Let us see that $\vec{\mathbf{v}}$ is an eigenvector of A with eigenvalue $\lambda \stackrel{\text{def}}{=} g(\vec{\mathbf{v}})$. By contradiction, suppose that $g(\vec{\mathbf{v}})\vec{\mathbf{v}} \neq A\vec{\mathbf{v}}$. By Lemma 2.7.11, $g(\vec{\mathbf{v}})\mathbf{v} \leq A\vec{\mathbf{v}}$ and $g(\vec{\mathbf{v}})\vec{\mathbf{v}} \neq A\vec{\mathbf{v}}$ imply

$$g(\vec{\mathbf{v}})A\vec{\mathbf{v}} = Ag(\vec{\mathbf{v}})\vec{\mathbf{v}} < AA\vec{\mathbf{v}}. \qquad 2.7.40$$

Since the inequality $g(\vec{\mathbf{v}})A\vec{\mathbf{v}} < AA\vec{\mathbf{v}}$ is strict, this contradicts the hypothesis that $\vec{\mathbf{v}}$ is an element of Q^* at which g achieves its maximum: $g(A\vec{\mathbf{v}})$ is the largest number such that $g(A\vec{\mathbf{v}})A\vec{\mathbf{v}} \leq AA\vec{\mathbf{v}}$, so $g(A\vec{\mathbf{v}}) > g(\vec{\mathbf{v}})$.

PAGE 229 Two mistakes in the first margin note. In $A\vec{\mathbf{w}}' = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \end{bmatrix}$, the $\mathbf{w}'$ should be $\mathbf{w}$.

Three lines further down, $g : (\vec{\mathbf{w}})\vec{\mathbf{w}}$ should be $g(\vec{\mathbf{w}})\vec{\mathbf{w}}$.

In the second margin note, $A\mathbf{w}/w_i$ should be $(A\mathbf{w})_i/w_i$.

PAGE 229 [added May 13, 2017] In inequality 2.7.42: $A\big(g(\vec{\mathbf{w}})\vec{\mathbf{w}}\big)$, not $A\big(g(\vec{\mathbf{w}})\big)\vec{\mathbf{w}}$.

PAGE 231 [added March 25, 2017] Exercise 2.7.4: The reference to Exercise 1.5.10 should begin this exercise, since the notation e^A is used in part a.

PAGE 242 [added May 22, 2017] In the second line of inequality 2.8.50, the term $4(u_2 - u_2)^2$ should be $4(u_2 - u_2')^2$.

PAGE 253 Three lines before inequality 2.9.6, "$\vec{\mathbf{h}}_n = |\mathbf{a}_{n+1} - \mathbf{a}_n|$" should be "$\vec{\mathbf{h}}_n = \mathbf{a}_{n+1} - \mathbf{a}_n$".

Page 280 Exercise 2.23: In the last line, "the nonzero columns of $\widetilde{B}$ form a basis for the kernel of A" should be "the columns of $\widetilde{B}$ corresponding to the zero columns of $\widetilde{A}$ form a basis for the kernel of A". We are also changing the notation to reflect the notation used in the solution, so A is $n \times m$, not $k \times n$, and I_n becomes I_m.

Page 285 Five lines before Definition 3.1.2: "a smooth k-dimensional manifold", not "a smooth n-dimensional manifold".

Page 292 [added May 22, 2017] Last margin note: The displayed equation should have $D_n F(\mathbf{z})$, not $D_n(\mathbf{z})$.

Page 312 First line of proof of Proposition 3.2.11: "Choose $\mathbf{x} \in U$", not "Choose $\mathbf{x} \in \mathbf{g}(U)$".

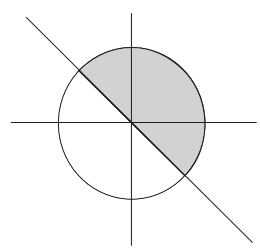
Page 312 Paragraph following formula 3.2.33: The compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ both make sense, but it is $\Psi \circ \Phi$, not $\Phi \circ \Psi$, that takes U to U. The domain of the derivative $[\mathbf{D}\Psi((\mathbf{y}, \mathbf{0}))]$ is $\mathbb{R}^m \times T_{\mathbf{x}}M^{\perp}$, not $T_{\mathbf{x}}M \times T_{\mathbf{x}}M^{\perp}$. In any eventual reprinting, we will rewrite this paragraph as follows:

The derivative $[\mathbf{D}\Psi((\mathbf{y}, \mathbf{0}))]$ is the map $\mathbb{R}^m \times T_{\mathbf{x}}M^{\perp} \to \mathbb{R}^n$ that maps $[\vec{\mathbf{a}}, \vec{\mathbf{b}}]$ to $[\mathbf{D}\gamma(\mathbf{y})]\vec{\mathbf{a}} + \vec{\mathbf{b}}$. It is an isomorphism because $[\mathbf{D}\gamma(\mathbf{y})] : \mathbb{R}^m \to T_{\mathbf{x}}M$ is an isomorphism. So there is a neighborhood U of $\mathbf{x}$ in $\mathbb{R}^n$ and a C^1 map $\Phi : U \to V \times T_{\mathbf{x}}M^{\perp}$ such that $\Psi \circ \Phi = \mathrm{id} : U \to U$. The composition of Φ with the projection onto V is our desired extension.

Page 365 [added August 1, 2017] Theorem 3.7.16: in two places in the third line, $\vec{\mathbf{v}}_i$ should be $\vec{\mathbf{v}}_1$:

$\ldots$ basis $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n$ of $\mathbb{R}^n$ with $A\vec{\mathbf{v}}_i = \lambda_i \mathbf{v}_i$, where $\lambda_1, \ldots, \lambda_k > 0$, $\ldots$

Page 366 Exercise 3.7.10 refers to a picture that was not included. Here it is:



Page 369 Definition 3.8.4, part 3:

"If $A \cap B = \emptyset$ for $A, B \subset S$", not "If $\mathbf{P}(A \cap B) = \emptyset$ for $A, B \subset S$".

Page 396 Exercise 3.9.10, part c: The formula for $H(x)$ is missing a 1/2; it should be

$$H(x) = \frac{1}{2(1 + (f'(x))^2)^{3/2}} \left(f''(x) - \frac{1 + (f'(x))^2}{f(x)} \right).$$

PAGE 400 Exercise 3.29, part a: "then A' is still symmetric" should be "then B is symmetric".

PAGE 438 [new, added May 22, 2018] Exercise 4.4.6 is incorrect; the rationals in $[0,1]$ are a counterexample. The exercise has been replaced by the following:

Show that if A$\subset \mathbb{R}^n$ has well-defined n-dimensional volume, then ∂A also has well-defined volume and $\mathrm{vol}_n\, \partial A = 0$.

PAGE 509 [added May 13, 2017] Second line: "by Theorem 4.11.4" should be "by part 3 of Proposition 4.1.14".

PAGE 513 [added, May 22, 2017] Equation 4.11.82: replace two dt by dx.

PAGE 528 [new, added May 22, 2018] Exercise 5.1.6 is incorrect as stated; vol_k is such a function V but it is not unique (for instance, the product of the lengths of the vectors is also such a function). The exercise and the margin note have been rewritten:

Exercise 5.1.6: We can think of V as a function of k vectors in $\mathbb{R}^n$: the columns of M. Multiplying by P on the left corresponds to rotating and reflecting these vectors (see Exercise 4.8.23). Multiplying by Q on the right is harder to visualize. It corresponds to rotating and reflecting the rows of M, i.e., the columns of $M^\top$.

5.1.6 Use the singular value decomposition (Theorem 3.8.1) to show that vol_k is the unique real-valued function V of $n \times k$ real matrices such that for all orthogonal $n \times n$ matrices P and all orthogonal $k \times k$ matrices Q, we have

$$V(M) = V(PMQ) \quad \text{and} \quad V(\sigma_1 \vec{\mathbf{e}}_1, \dots, \sigma_k \vec{\mathbf{e}}_k) = |\sigma_1 \cdots \sigma_k|$$

for any $n \times k$ real matrix M and any numbers $\sigma_1, \dots, \sigma_k$.

PAGE 624 [added May 13, 2017] Exercise 6.6.2: By definition, a piece-with-boundary must be compact, but the loci in parts a and b are not compact. We are adding to each the additional constraint $x^2 + y^2 \leq 1$.

PAGE 639 [added May 13, 2017] Exercise 6.8.7: "For what vector field $\vec{F}$ can φ be written $W_{\vec{F}}$?", not "For what vector field $\vec{F}$ can φ be written $dW_{\vec{F}}$?" (φ is a 1-form and dW_F is a 2-form, so the original wording did not make sense).

PAGE 653 [added May 13, 2017] Proposition 6.10.8: W is defined as a "bounded open subset of $\mathbb{R}^k$", but since W is open, it's not clear that it makes sense to "give $\partial(W \cap Z)$ the boundary orientation". Instead we should write "let W be a piece-with-boundary of $\mathbb{R}^k$", and in the 4th line, change "with compact support in W to "with compact support in the interior of W". Further, in the caption to Figure 6.10.6, " Since $\varphi = 0$ outside W" should be "Since $\varphi = 0$ on ∂W".

PAGE 727 First line after equation A7.10: "When $|\mathbf{y}_0 + \vec{\mathbf{k}}| \in V$" should be "When the point $\mathbf{y}_0 + \vec{\mathbf{k}}$ is in V".

Page 802 First line after A23.8: "if", not "if and only if": this matrix is surjective if at least one . . .

Minor typos

Page 25 Line 1: There should be no "s" between "consisting" and "alternately".

Page 38 [added September 4, 2017] In the second line of the second paragraph, "because haven't defined" should be "because we haven't defined".

Page 101 In the last line before Corollary 1.5.40, $(1 - A)$ should be $(I - A)$.

Page 109 [added May 27, 2017] Two lines before equation 1.6.9: Delete "is": "Next we want to show that f has a minimum."

Page 137 In the second line of Section 1.8, there should be no "s" after "given" (the extraneous s's no doubt come from an attempt to save the file).

Page 214 Second margin note, line 5: "subspace $E \subset \mathbb{R}^n$", not "subset $E \subset \mathbb{R}^n$".

Page 227 [added Sept. 30] Page 227: The last margin note should start "If A is a real matrix", not "If A s a real matrix".

Page 296 [added July 4, 2017] Two lines before Theorem 3.1.16: "an arbitrary C^1 mapping", not "an arbitrary C^1 mappings".

Page 311 Two lines before equation 3.2.31, the x in $T_x M$ should be bold: $T_{\mathbf{x}} M$.

Page 316 Table 3.3.2: The 2x in the monomial row should be $2x$ (x in math mode)

Page 318 [added July 4, 2017] In the title of Theorem 3.3.8, the left parenthesis is missing.

Page 340 [added May 22, 2017] Three lines after equation 3.5.32: "identity" should be "identify".

Page 359 [added August 1, 2017] There is a misplaced end parenthesis in equation 3.7.39. The middle term should be $= [\mathbf{D}f(\gamma(\mathbf{v}_0))][\mathbf{D}\gamma(\mathbf{v}_0)] =$

Page 376 [added May 22, 2017] Caption to Figure 3.8.6: "face" should be "eigenface". In the last line, "egenvalue" should be "eigenvalue".

Page 420 [added August 1, 2017] Definition 4.2.6: “probability density”, not “probability of density”. “If f is”, not “if f be”.

Equation 4.2.13: The middle expression for the covariance is missing a set of parentheses. It should be

$$E\Big(\big(f - E(f)\big)\big(g - E(g)\big)\Big).$$

Page 427 Formula 4.3.8 is missing a closing parenthesis on the right: $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| < \epsilon$, not $|f(\mathbf{x}_1) - f(\mathbf{x}_2| < \epsilon$

Page 451 [added May 13, 2017] In Table 4.6.3, the word “Function” should be in the first row, not the second.

Page 488 [added Aug. 10, 2017] In the first margin note, to be consistent with equation 4.10.4, “the r in $r\,dr\,d\theta$ plays the role” should be “the r in $r\,|dr\,d\theta|$ plays the role”.

Page 510 [added Aug. 12, 2017] In equation 4.11.67, on the right side, there is an extra end parenthesis: $(f \circ \Phi)(\mathbf{u})\big)$ should be $(f \circ \Phi)(\mathbf{u})$.

Page 534 [added May 13, 2017] Line immediately before formula 5.2.15: we should have written $M \subset \mathbb{R}^n$, not just M.

Page 584 [added May 22, 2017] We forgot parentheses in the third line of Definition 6.3.3: $\mathcal{B}_{\mathbf{x}}(M)$, not $\mathcal{B}_{\mathbf{x}}M$.

Page 588 [added May 22, 2017] Exercise 6.3.13, part a: The first displayed equation is missing an arrow on $\mathbf{e}_1$.

Page 597 [added May 22, 2017] First line: “over the piece”, not “through the piece”. In the first line of equation 6.4.33, the two column vectors should be in parentheses:

$$P_{\begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix}}\left(\overbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}^{\overrightarrow{D_1\gamma}}, \overbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}^{\overrightarrow{D_2\gamma}}\right)$$

Page 599 [added May 13, 2017] The first margin note is missing an end parenthesis: “$\omega(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$” should be “$\omega(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k))$”.

Page 612 [added May 22, 2017] Line 4: “require”, not “rquire”.

Page 614 [added May 13, 2017] Last line: “obviously”, not “obvioiusly”.

Page 705 First margin note: equation A1.2, not equation AA1.2

PAGE 719 Three lines before the bottom: "□ Lemma A5.3" indicating the end of the proof does not belong here. Inequality A5.26 should end with a period, not a comma.

PAGE 726 Last line: the left side of equation A7.8 should be $|\mathbf{g}(\mathbf{y}_1) - \mathbf{g}(\mathbf{y}_2)|$, not $|\mathbf{g}(\mathbf{y}_1)| - \mathbf{g}(\mathbf{y}_2)|$.

PAGE 733 [added May 22, 2017] Some standard basis vectors in equations A9.5, A9.6, and A9.7 are missing arrows: $\mathbf{e}_i$ should be $\vec{\mathbf{e}}_i$ and $\mathbf{e}_j$ should be $\vec{\mathbf{e}}_j$.

PAGE 811 [added May 22, 2017] The entry page 625 for the fundamental theorem of calculus should be 626.

Notes and clarifications

PAGES 41-42 Exercise 1.1.8 could have been clearer. On page 41, "where a is" could be replaced by "the function a gives". On page 42, part b might include "The pipe forms a torus, with $r \leq 1$".

PAGE 79 [added September 13, 2017] We prove part 3 of Proposition 1.4.19 for $\det[\vec{\mathbf{a}} \times \vec{\mathbf{b}}, \vec{\mathbf{a}}, \vec{\mathbf{b}}]$, but by the margin note on page 78, this is the same as proving it for $\det[\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{a}} \times \vec{\mathbf{b}}]$. Using the language of antisymmetry (see Theorem and Definition 4.8.1), going from one to the other requires two exchanges of arguments, so the sign changes twice and remains the same.

PAGE 79 In the proof of Proposition 1.4.21, θ is not well defined; there are two angles between a line l and a vector anchored at a point of l: one in $[0, \pi/2]$, the other in $[\pi/2, \pi]$. We want $\theta \in [0, \pi/2]$ to get $\cos\theta \geq 0$, so that the volume won't be negative.

PAGES 91 AND 92 [added September 4, 2017] We have added some comments about our definition of limits. At the bottom of page 91, next to equation 1.5.24, we are adding the margin note "If $\mathbf{x}_0$ is in the domain of f, one way to "approach" $\mathbf{x}_0$ is to *be* $\mathbf{x}_0$."

On page 92, in the margin, we have added some terminology. The standard U.S. definition is sometimes called the *deleted limit*; the definition we have adopted is sometimes called the *undeleted limit*.

PAGE 115 Equation 1.6.28 would perhaps be better with "dog" in an underbrace:

$$q(u) = \overbrace{\underbrace{b_0}_{\text{flagpole}} + b_j u^j}^{\text{position of man}} + \underbrace{b_{j+1}u^{j+1} + \cdots + u^k}_{\text{leash}} = \underbrace{p(z)}_{\text{dog}}. \qquad 1.6.28$$

PAGE 132 [added May 27, 2017] First line of the footnote: We will replace "a vector H" by "the increment H; this "vector" is an $n \times n$ matrix." Vector spaces that are not naturally $\mathbb{R}^n$ are discussed in Section 2.6.

Page 170 Corollary 2.2.7 also uses Proposition 1.3.13; the proof of that proposition shows that if the linear transformation "multiplication by A" is bijective, its inverse mapping is linear and so has a matrix; this matrix is a two-sided inverse of A.

Page 171 The paragraph following formula 2.2.11 was not clear: the antecedent of "it" in the second line was not specified, and in the next-to-last line we did not specify that $\vec{\mathbf{x}}' \neq \vec{\mathbf{x}}$. We propose instead:

> Next note that if B is a right inverse of A, i.e., $AB\vec{\mathbf{a}} = \vec{\mathbf{a}}$ for every $\vec{\mathbf{a}}$, then A is onto, since $A\vec{\mathbf{x}} = \vec{\mathbf{a}}$ has a solution for every $\vec{\mathbf{a}}$, namely $\vec{\mathbf{x}} = B\vec{\mathbf{a}}$. To see that if B is a left inverse of A then A must be one to one, suppose two vectors $\vec{\mathbf{x}}'$ and $\vec{\mathbf{x}}''$ satisfy $A\vec{\mathbf{x}}' = A\vec{\mathbf{x}}'' = \vec{\mathbf{a}}$. Then $\vec{\mathbf{x}}' = BA\vec{\mathbf{x}}' = B\vec{\mathbf{a}}$ and $\vec{\mathbf{x}}'' = BA\vec{\mathbf{x}}'' = B\vec{\mathbf{a}}$, so $\vec{\mathbf{x}}' = \vec{\mathbf{x}}''$. Therefore, by formula 2.2.11, if A is square and $AB = I$, then $BA = I$.

We would also like to expand on the second margin note, making it

> The paragraph following formula 2.2.11 has nothing to do with linearity: For any map $f : X \to Y$, being onto corresponds to having a right inverse and being one to one corresponds to having a left inverse.

Page 214 [added September 15, 2017] At the end of the first margin note we plan to add:

Note that multiplying both sides on the right by $[P_{\mathbf{v}'\to\mathbf{v}}]^{-1}$ gives

$$[\mathbf{v}_1 \dots \mathbf{v}_n] = [\mathbf{v}'_1 \dots \mathbf{v}'_n][P_{\mathbf{v}'\to\mathbf{v}}]^{-1};$$

the change of basis matrix from $\{\mathbf{v}'\}$ to $\{\mathbf{v}\}$ is the inverse of the change of basis matrix from $\{\mathbf{v}\}$ to $\{\mathbf{v}'\}$:

$$[P_{\mathbf{v}\to\mathbf{v}'}] = [P_{\mathbf{v}'\to\mathbf{v}}]^{-1}.$$

Page 229 [added March 22, 2017, with the addition of P in "the plane spanned" added in May] We have added the following margin note, to address the issue why (2 lines after equation 2.7.41), "$\vec{\mathbf{v}}$ is the unique eigenvector of A in Δ":

"The plane P spanned by $\vec{\mathbf{v}}$ and $\vec{\mathbf{v}}'$ necessarily intersects ∂Q, and $A = \lambda \mathrm{id}$ on P means that $\partial Q \cap P$ is mapped to $\partial Q \cap P$, not to $\mathring{Q}$."

Note that the notation id for the identity transformation was introduced in Example 1.3.6.

Page 311 Proposition and Definition 3.2.9: We used the notation for a restricted function before we explain it on page 359. Here, $[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]|_{T_\mathbf{x}M}$ denotes the linear function $[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})] : \mathbb{R}^n \to \mathbb{R}^k$ restricted to $T_\mathbf{x}M \subset \mathbb{R}^n$.

Page 311 Proof of Proposition 3.2.10: What justifies our saying (two lines before equation 3.2.31) that $\ker[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]$ and $T_\mathbf{x}M$ span $\mathbb{R}^n$?

Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$; then $[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]\mathbf{u} \in \mathbb{R}^k$. Since $[\mathbf{Df}(\mathbf{x})]$ is onto $\mathbb{R}^k$, there exists $\mathbf{u}_1 \in T_{\mathbf{x}}M$ such that

$$[\mathbf{Df}(\mathbf{x})]\mathbf{u}_1 = [\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]\mathbf{u}.$$

By equation 3.2.27,

$$[\mathbf{Df}(\mathbf{x})]\mathbf{u}_1 = [\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]\mathbf{u}_1,$$

so $[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]\mathbf{u} = [\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]\mathbf{u}_1$, i.e.,

$$\mathbf{u} - \mathbf{u}_1 \in \ker[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})].$$

Therefore we can write $\mathbf{u} \in \mathbb{R}^n$ as $\mathbf{u} = (\mathbf{u} - \mathbf{u}_1) + \mathbf{u}_1$, with $\mathbf{u}_1 \in T_{\mathbf{x}}M$ and $\mathbf{u} - \mathbf{u}_1 \in \ker[\mathbf{D}\widetilde{\mathbf{f}}(\mathbf{x})]$.

Page 352 Theorem and Definition 3.7.5: We neglected to specify how the C^1 mapping $\mathbf{F}$ defines X; it defines it by the equation

$$X \stackrel{\text{def}}{=} \mathbf{F}^{-1}(\mathbf{0}).$$

Note that since adding a constant to $\mathbf{F}$ does not change its derivative, the theorem is true for $X = \mathbf{F}^{-1}(\mathbf{b})$ for any $\mathbf{b}$. We use this, for example, in equation 3.7.62, when we write $F_1(\vec{\mathbf{x}}) = 1$ rather than $F_1(\vec{\mathbf{x}}) - 1 = 0$.

Page 451 [added May 13, 2017] We are adding as a margin note: Proof of part 1: The change of variables

$$x \mapsto \frac{1}{2}\big((1+x)b + (1-x)a\big)$$

will bring an integral over any segment $[a, b]$ to an integral over $[-1, 1]$; when integrating a polynomial this does not change the degree of the polynomial.

Page 461 [added May 22, 2017] First paragraph of Section 4.8: It would probably be better to write "gives the signed area" and "gives the signed volume", and to replace "geometric interpretation, as a signed volume, by "geometric interpretation as a signed n-dimensional volume".

Page 502 [added Sept. 14, 2017] We have changed definition 4.11.8 to include an explicit definition of "Lebesgue-integrable function":

Definition 4.11.8 (Lebesgue-integrable function, Lebesgue integral). Let $k \mapsto f_k$ be a sequence of R-integrable functions such that

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\mathbf{x})||d^n\mathbf{x}| < \infty. \qquad 4.11.19$$

Then $f \underset{L}{=} \sum_{k=1}^{\infty} f_k$ is *Lebesgue integrable*, and its *Lebesgue integral* is

$$\int_{\mathbb{R}^n} f(\mathbf{x})|d^n\mathbf{x}| \overset{\text{def}}{=} \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(\mathbf{x})|d^n\mathbf{x}|. \qquad 4.11.20$$

PAGE 527 [added May 22, 2017] The notion of a vector anchored at a point is discussed in Section 1.1. We might spell out what is meant here: "We denote by $P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ a k-parallelogram in $\mathbb{R}^n$ anchored at $\mathbf{x} \in \mathbb{R}^n$: the k vectors spanning the parallelogram all begin at $\mathbf{x}$."

PAGE 529 [added May 13, 2017] Definition 5.2.3: We are expanding part 4, which is now "the derivative $[D\gamma(\mathbf{u})]$ is one to one for all $\mathbf{u}$ in $U - X$, with image $T_{\gamma(\mathbf{u})}M$"

We are also adding a margin note:

"Part 4: Recall from Proposition 3.2.7 that $T_{\gamma(\mathbf{u})}M = \operatorname{img}[\mathbf{D}\gamma(\mathbf{u})]$."

PAGE 575 [added May 22, 2017] Equation 6.1.40 has $\mathbf{x}$ on the right, no $\mathbf{x}$ on the left. We will take out the $\mathbf{x}$ on the right and elaborate, writing

$$\varphi = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1,\dots,i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}; \qquad 6.1.40$$

for each $i_1, \dots, i_k$ satisfying $1 \le i_1 < \cdots < i_k \le n$, the map $\mathbf{x} \mapsto a_{i_1,\dots,i_k}(\mathbf{x})$ is a real-valued function on U.

PAGE 585 [added May 22, 2017] At the end of Example 6.3.6, we will add a reference to concentric circles: " . . . is a nonvanishing vector field tangent to the circle of equation $x^2 + y^2 = R^2$ (and to all concentric circles), defining the *counterclockwise* orientation."

PAGE 587 [added May 13, 2017] Proof of Proposition 6.3.10: We are replacing the second paragraph by:

We must show that if M is connected and Ω' and Ω'' are two orientations of M such that $\Omega'_{\mathbf{x}_0} = \Omega''_{\mathbf{x}_0}$ for one point $\mathbf{x}_0 \in M$, then $\Omega'_{\mathbf{x}} = \Omega''_{\mathbf{x}}$ for all $\mathbf{x} \in M$.

Choose a path $\gamma : [a, b] \to M$ with $\gamma(a) = \mathbf{x}_0$ and $\gamma(b) = \mathbf{x}$. For all t there exists $s(t) \in \{+1, -1\}$ such that $\Omega'_{\gamma(t)} = s(t)\Omega''_{\gamma(t)}$. If s is not constant, there exists a sequence $t_n \to t_\infty$ in $[a, b]$ such that $s(t_n) = -1$

for all n, but $s(t_\infty) = +1$. Choose a basis $\{\mathbf{b}\}$ of $T_{\gamma(t_\infty)}M$ such that $\Omega'(\gamma(t_\infty), \{\mathbf{b}\}) = \Omega''(\gamma(t_\infty), \{\mathbf{b}\})$. By Exercise 6.3.16 (see note for page 589) there exist bases $\{\mathbf{b}_n\}$ of $T_{\gamma(t_n)}M$ such that $(\gamma(t_n), \{\mathbf{b}_n\})$ converges to $(\gamma(t_\infty), \{\mathbf{b}\})$. Then

$$s(t_n) = \frac{\Omega'(\gamma(t_n), \{\mathbf{b}_n\})}{\Omega''(\gamma(t_n), \{\mathbf{b}_n\})} = -1 \quad \text{but} \quad \frac{\Omega'(\gamma(t_\infty), \{\mathbf{b}\})}{\Omega''(\gamma(t_\infty), \{\mathbf{b}\})} = +1.$$

But Ω'/Ω'' is continuous (Theorem 1.5.29), so s is constant. □

Page 589 [added May 13, 2017] Exercise 6.3.16 is new, as is the hint in the margin.

6.3.16 Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold in $\mathbb{R}^n$. Prove that $\mathcal{B}(M)$ is a manifold in $\mathbb{R}^{n(k+1)}$, and that every open subset of $\mathcal{B}(M)$ projects to an open subset of M. See the hint in the margin.

Hint for Exercise 6.3.16: Denote by $\mathrm{GL_k}\mathbb{R}$ the set of invertible $k \times k$ real matrices (which is open in $\operatorname{Mat}(n,n)$ by Corollary1.5.40). Suppose that U is a subset of $\mathbb{R}^k$ and $\gamma : U \to \mathbb{R}^n$ is a parametrization of an open subset of M. Show that the map

$$\Gamma : U \times \mathrm{GL_k}\,\mathbb{R} \to \mathbb{R}^{n(k+1)}$$

defined by

$$\Gamma(\mathbf{u}, [\mathbf{a}_1, \dots, \mathbf{a}_n] =$$
$$\Big(\gamma(\mathbf{u}), [\mathbf{D}\gamma(\mathbf{u})]\mathbf{a}_1, \dots [\mathbf{D}\gamma(\mathbf{u})]\mathbf{a}_k\Big)$$

parametrizes an open subset of $\mathcal{B}(M)$.