9
Dynamics of polynomials

Chapters 9 and 10 concern dynamical systems. Dynamical systems come in two flavors: continuous time (differential equations) and discrete time (iterations). Here we are concerned with discrete time. We take a map $f: X \to X$ (it is essential that the domain and codomain be the same) and we consider sequences of iterates $x, f(x), f(f(x)), \dots$, which we think of as time 0, time 1, time 2, $\dots$. We denote the nth iterate by $f^{\circ n}(x)$.[1] The study of these dynamical systems essentially consists of trying to describe the behavior of such sequences.

If $f: X \to X$ and $g: Y \to Y$ are two dynamical systems and $\varphi: X \to Y$ satisfies $\varphi \circ f = g \circ \varphi$, then for all n,

$$\varphi \circ f^{\circ n} = g^{\circ n} \circ \varphi. \tag{9.0.1}$$

This is a trivial but important application of associativity of composition: for the first step in the iteration,

$$\varphi \circ f^{\circ 2} = (\varphi \circ f) \circ f = g \circ (\varphi \circ f) = g \circ g \circ \varphi = g^{\circ 2} \circ \varphi. \tag{9.0.2}$$

Thus the sequences $n \mapsto f^{\circ n}(x)$ and $n \mapsto g^{\circ n}\big(\varphi(x)\big)$ correspond under φ. Such maps φ are the *morphisms* of dynamical systems, since they preserve sequences of iterates. These morphisms are called *semi-conjugacies*; when φ is invertible, then $\varphi \circ f = g \circ \varphi$ can be written $f = \varphi^{-1} \circ g \circ \varphi$ and φ is a *conjugacy* between f and g.

Conjugacy is the notion of isomorphism for dynamical systems; dynamical systems is all about establishing conjugacies between complicated maps and simpler ones.

In this chapter we give an introduction to the dynamics of polynomials $p: \mathbb{C} \to \mathbb{C}$. As soon as a polynomial has degree greater than 1, its iterates become tremendously complicated. We won't be able to find global conjugacies with simpler functions, but we will find local conjugacies and local semi-conjugacies; examples include Theorems 9.1.6, 9.2.8, 9.2.12, 9.2.20, and 9.4.1. These results will be used in Section 10.5 in the context of quadratic polynomials, but here we avoid focusing on the quadratic case. Many of the results (including all the local theory) are true for rational functions, entire functions, and even functions meromorphic on $\mathbb{C}$, but to keep the chapter of reasonable length we have avoided that generality also.

[1]If $Y \subset X$, then $f^{-1}(Y)$ is the inverse image of Y under f; this is always defined and never ambiguous. When f is invertible we will also write f^{-1} and f^{-n} for $f^{\circ(-1)}$ and $f^{\circ(-n)}$ and hope this will create no confusion with $1/f$.

9.1 Julia sets

In complex dynamics in one variable the fundamental object of study is the *Julia set*.

Definition 9.1.1 (Julia set, filled Julia set) Let p be a polynomial of degree $d \geq 2$. The *Julia set* J_p and the *filled Julia set* K_p are

$$K_p := \left\{ z \in \mathbb{C} \;\middle|\; \text{the sequence } z, p(z), p^{\circ 2}(z), \dots \text{ is bounded} \right\}$$
$$J_p := \partial K_p$$

The sequence $n \mapsto p^{\circ n}(z)$ is called the *orbit* of z under p. The complement of the Julia set is often called the *Fatou set*.

It is distinctly easier to study the Julia set of a polynomial than the Julia set of a more general mapping, because the Julia set of a polynomial is the boundary of the filled Julia set. Rational functions that are not polynomials have a Julia set but no filled Julia set.

Figure 9.1.1 represents six filled Julia sets K_p, colored black, for appropriate cubic polynomials. We chose $d = 3$ rather than $d = 2$ to avoid overlap with Chapter 10, and because these exhibit a larger variety of behavior.

Each picture represents hundreds of millions of multiplications. How were the pictures made? We began by choosing some number R, large compared to the coefficients of the polynomial (in this case, $R = 100$ but a smaller R would have done). The colors represent "rates of escape": if the orbit $n \mapsto p^{\circ n}(z)$ tends to infinity, there is a first n for which $|p^{\circ n}(z)| > R$; each dot is colored according to the number n. The picture will not depend in any essential way on R. If $|f^{\circ n}(z)| > 10$, then $|f^{\circ(n+1)}(z)| \sim 1000$ and $|f^{\circ(n+2)}(z)| \sim 10^9$. When a point is well in the escape region, it really goes!

A more crucial issue is when to color a pixel black: how many iterations do we make before giving up and deciding that a point will never escape? How do we choose N such that if $|f^{\circ N}(z)| < R$, we decide that $z \in K_f$? Figure 9.1.1 was made with $N = 1000$; different choices of N can produce quite different pictures, at very different computational cost.

Proposition 9.1.2 *The sets K_p and J_p are compact subsets of $\mathbb{C}$.*

Proof Set $p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0$ with $a_d \neq 0$, and set

$$A = \max\{(|a_{d-1}| + \cdots + |a_0|), 1\} \qquad 9.1.1$$

Set $R = \max\left\{\frac{2+A}{|a_d|}, 1\right\}$. Then if $|z| \geq R$

$$\begin{aligned}|p(z)| &\geq |a_d||z|^d - \left(|a_{d-1}||z^{d-1}| + \cdots + |a_0|\right)\\ &\geq |a_d||z|^d - A|z|^{d-1} \qquad 9.1.2\\ &\geq |z|^{d-1}(|a_d||z| - A) \geq 2|z|.\end{aligned}$$

Thus $K_p \subset \{z \in \mathbb{C} \mid |z| < R\}$, so K_p is bounded, and

$$K_p = \bigcap_{n=1}^{\infty} p^{-n}\big(\overline{D}_R(0)\big), \qquad 9.1.3$$

so it is an intersection of closed sets, hence closed. The boundary of a compact subset is always compact, so J_p is compact. □

REMARK in the proof of Proposition 9.1.2, the 2 in $\frac{2+A}{|a_d|}$ doesn't have to be 2; any number > 1 will do. △

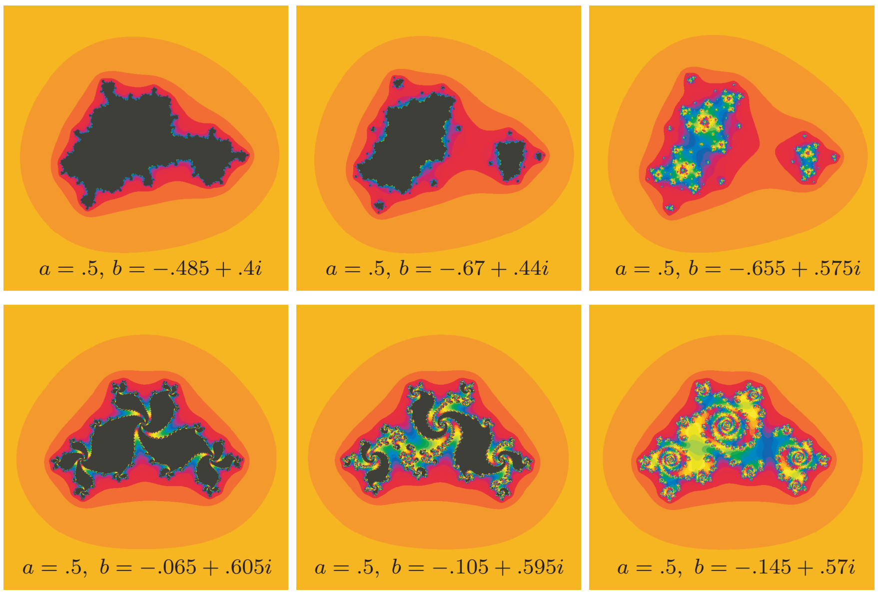

FIGURE 9.1.1 Two sets of three pictures of filled Julia sets for cubic polynomials, all written in the form $p(z) = z^3 - 3a^2z + b$, for $|\operatorname{Re} z| \le 2$, $|\operatorname{Im} z| \le 2$. The pictures in each row correspond to polynomials whose coefficients are quite close, although the filled Julia sets are very different. The sets K_p are black; the colors represent "rate of escape to infinity". These pictures also illustrate Theorem 9.1.6, which says that the behavior of critical points under iteration is essential to understanding the sets K_p. The critical points of p are $\pm a = \pm 1/2$. In the left pictures both critical points have bounded orbits, so the filled Julia sets K_p are connected. In both middle pictures one critical point has a bounded orbit and the other not; the sets K_p are not connected, but some components of K_p are not points (and are themselves fairly complicated in the bottom center picture). For the polynomials pictured at right, both critical points escape, and $K_p = J_p$ is a Cantor set.

If f is a rational function or an entire function, the Julia set of f is defined as the set of non-normality of the sequence $f, f^{\circ 2}, f^{\circ 3}, \dots$. Proposition 9.1.3

shows that the definitions are consistent: for polynomials, the boundary of the filled Julia set is indeed the set of non-normality. In this context, a sequence of functions that converges locally uniformly to infinity is normal, i.e., normal as a sequence of functions with values in $\mathbb{P}^1$.

Proposition 9.1.3 *Let p be a polynomial of degree $d \geq 2$. Then $z \notin J_p$ if and only if z has a neighborhood on which the sequence $p, p^{\circ 2}, p^{\circ 3}, \dots$ is normal.*

PROOF If $z \notin J_p$, then either z is in the interior of K_p or z is not in K_p. In the first case, z has a neighborhood $U \subset \overset{\circ}{K}_p$ and the sequence $n \mapsto p^{\circ n}$ is bounded on U, hence normal. In the second case, there is some N such that $|p^{\circ N}(z)| > R$ (where R is as in the proof of Proposition 9.1.2). There is then a neighborhood U of z such that $|p^{\circ N}(w)| > R$ for all $w \in U$. Since for all $n \geq N$ we then have $|p^{\circ n+1}(w)| > 2|p^{\circ n}(w)|$, the sequence $n \mapsto p^{\circ n}$ converges uniformly to ∞ on U.

Conversely, if $z \in J_p$, then every neighborhood of z contains points with bounded orbits (including z itself), and open subsets on which the sequence $n \mapsto p^{\circ n}$ converges to infinity, so the sequence has no subsequence converging uniformly on compact subsets. □

Proposition 9.1.4 *1. For any polynomial p of degree $d \geq 2$, both K_p and J_p are totally invariant:*

$$p(J_p) = J_p, \quad p^{-1}(J_p) = J_p, \quad p(K_p) = K_p, \quad p^{-1}(K_p) = K_p.$$

2. Conversely, if $X \subset \mathbb{C}$ is a closed set satisfying $p(X) = p^{-1}(X) = X$, then either

a. $X = \{x\}$ is a single point, p is conjugate to $z \mapsto z^d$ and x corresponds to 0 under the conjugacy,

or

b. $J_p \subset X$.

PROOF 1. The first part is obvious: a point z has a bounded orbit under p if and only if the orbit of $p(z)$ is bounded, which is true if and only if for each $z_1 \in p^{-1}(z)$ the orbit of z_1 is bounded.

2. The second part requires Montel's theorem. If X contains two or more points, the sequence $n \mapsto p^{\circ n}$ of iterates omits these points, hence is normal by Montel's theorem, so J_p is a subset of X.

If $X = \{x\}$ is a singleton, then x is a fixed point of p, and is the only solution of $p(z) = x$. Thus $p(z) = b(z-x)^d + x$ for some b, i.e., p is conjugate to $Z \mapsto bZ^d$. A further conjugacy, setting $Z = W\eta$ with $\eta^{d-1} = 1/b$, makes p conjugate to $W \mapsto W^d$. □