8

The classification of homeomorphisms of surfaces

In this chapter we present and prove the first of Thurston's theorems involving Teichmüller spaces: Theorem 8.1.4, which classifies homeomorphisms of surfaces into three types: periodic, reducible, and pseudo-Anosov.

Understanding homeomorphisms and diffeomorphisms of manifolds is a central problem of mathematics. Even understanding homeomorphisms and diffeomorphisms of the circle is an immensely difficult problem with a huge literature. The 2-dimensional case is much harder yet; Thurston's theorem is probably the main result in the field. The theorem concerns homeomorphisms up to homotopy, so it is in some sense crude, avoiding all delicate local study; in exchange, it provides vital global information. Moreover, the group of homotopy classes of homeomorphisms, also known as the *mapping class group,* is of central interest in geometric group theory. Here also Thurston's theorem is of fundamental importance.

Section 8.1 states the classification theorem and defines the types of homeomorphisms. Section 8.2 gives examples of periodic and reducible homeomorphisms. Section 8.3 describes two families of pseudo-Anosov homeomorphisms in addition to the Arnoux-Yoccoz homeomorphism. Section 8.4 proves the classification theorem. Section 8.5 studies homeomorphisms with marked points. Finally, Section 8.6 explores what numbers can be *stretch factors* of pseudo-Anosov homeomorphisms.

We will present a proof of the classification theorem due to Bers [6]; it is more in keeping with the style of this book than Thurston's proof. Thurston's proof has been given in considerable detail by Fathi, Laudenbach, and Poenaru [31]; it is much longer and more elaborate.[1]

Thurston's terminology was inspired by the classification of homeomorphisms of the torus.

Classification of homeomorphisms of the torus

Let T denote the torus $T := \mathbb{R}^2/\mathbb{Z}^2$. A matrix $A \in \mathrm{SL}_2\,\mathbb{Z}$ induces a map $\mathbb{R}^2 \to \mathbb{R}^2$, which induces an orientation-preserving homeomorphism $f_A : T \to T$ since $A\mathbb{Z}^2 = \mathbb{Z}^2$.

[1] Apparently Jakob Nielsen has some claim to having proved the result long before Thurston. However, I have spoken with the people who know Nielsen's work best, and they say that he never made any definition similar to "pseudo-Anosov". Without it, no classification theorem seems possible.

Conversely, the parametrized closed curves

$$t \mapsto \begin{pmatrix} t \\ 0 \end{pmatrix},\ 0 \le t \le 1 \quad \text{and} \quad t \mapsto \begin{pmatrix} 0 \\ t \end{pmatrix},\ 0 \le t \le 1$$

form a basis of the homology group $H_1(T;\mathbb{Z}) = \mathbb{Z}^2$, and any orientation-preserving homeomorphism $f : T \to T$ gives a homomorphism $f_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ that has a matrix $A \in \mathrm{SL}_2\,\mathbb{Z}$. Lift f to $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$; the straight line segments joining $\widetilde{f}\begin{pmatrix} x \\ y \end{pmatrix}$ to $\widetilde{f}_A\begin{pmatrix} x \\ y \end{pmatrix}$ descend to give a homotopy between f and f_A, and then Theorem C3.1 shows that f and f_A are isotopic.

Thus the classification of homeomorphisms of T up to isotopy is the same as the classification of elements of $\mathrm{SL}_2\,\mathbb{Z}$.

This classification leads to three cases: a matrix $A \in \mathrm{SL}_2\,\mathbb{Z}$ may have complex nonreal eigenvalues, a double eigenvalue ± 1, or real distinct eigenvalues. The eigenvalues of A are the roots of $\lambda^2 - (\operatorname{tr} A)\lambda + 1 = 0$. They can be nonreal if and only if $\operatorname{tr} A = 0$ or $\operatorname{tr} A = \pm 1$ (remember that the trace $\operatorname{tr} A$ is an integer). If $\operatorname{tr} A = 0$, the matrix has eigenvalues $\pm i$, and $A^4 = I$. If $\operatorname{tr} A = -1$, then $A^3 = I$, and if $\operatorname{tr} A = 1$, then $A^6 = I$.

If $\operatorname{tr} A = \pm 2$, then ± 1 is an eigenvalue, and a corresponding eigenvector provides a simple closed curve on the surface that is mapped to itself (preserving or reversing the orientation).

If $|\operatorname{tr} A| > 2$, then A has two distinct real eigenvalues, necessarily irrational: the *contracting* eigenvalue $|\lambda_1| < 1$ and the *expanding* eigenvalue $|\lambda_2| > 1$. The directions of the eigenvectors provide invariant foliations on $\mathbb{R}^2/\mathbb{Z}^2$, which are contracted and expanded by f_A. These homeomorphisms f_A are called *Anosov*; in [1], Anosov studied them and in particular showed that they are structurally stable.

8.1 The classification theorem

Thurston's classification theorem is an analogue of the classification of homeomorphisms of tori; it applies to surfaces of any genus $g \ge 2$. Anosov homeomorphisms are replaced by *pseudo-Anosov homeomorphisms*, which also have invariant foliations that are expanded and contracted, as shown in Figure 8.1.1. The leaves of these foliations are the horizontal and vertical trajectories of a quadratic differential q. Unlike the case of the torus, however, the foliations are singular. Examples of such singularities are shown in Figure 5.3.1 in Volume 1.

Definition 8.1.1 (Types of homeomorphisms of surfaces) Let S be a compact surface of genus $g \ge 2$, and let $f : S \to S$ be an orientation-preserving homeomorphism. The homeomorphism f is

1. *periodic* if the iterate $f^{\circ m}$ is the identity for some $m \geq 1$
2. *reducible* if some nonempty multicurve is invariant under f (such a multicurve is called a *reducing multicurve*)
3. *pseudo-Anosov* if there exist an element $\varphi\colon S \to X$ of Teichmüller space $\mathcal{T}_S$, a holomorphic quadratic differential $q \in Q(X)$, and $K > 1$ such that $\varphi \circ f \circ \varphi^{-1}$ is a Teichmüller mapping
$$(X, q) \to (X, q/K)$$

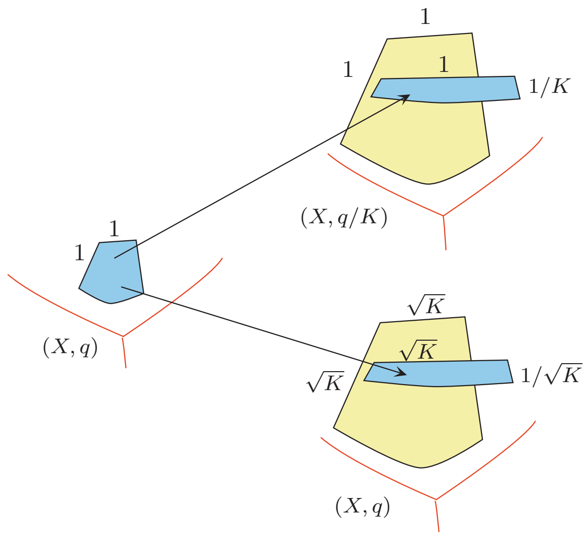

FIGURE 8.1.1 LEFT: A piece of a Riemann surface X with quadratic differential q, and a (blue) unit square in the natural coordinate for q. The two blue regions at right are the image of the unit square at left by the same map f (strictly speaking, by $\varphi \circ f \circ \varphi^{-1}$). The two pictures on the right are identical, but with different metrics. BOTTOM RIGHT: A pseudo-Anosov homeomorphism from (X, q) to (X, q); it stretches horizontal segments of curves by the *stretch factor* $\lambda = \sqrt{K}$, and shrinks vertical segments by the same factor, preserving area. TOP RIGHT: The same map seen as a Teichmüller map $(X, q) \to (X, q/K)$ that maps horizontal segments to horizontal segments of the same length, and shrinks vertical segments by a factor of K. (Since the metric at upper right is smaller than that at left, the (yellow) unit square at right is of course larger.)

REMARK As suggested by Figure 8.1.1, it is usually better to think of a pseudo-Anosov homeomorphism as an area-preserving homeomorphism $f\colon (X, q) \to (X, q)$ rather than as a Teichmüller map $f\colon (X, q) \to (X, q/K)$.

Suppose $s_2 = f(s_1)$ and $q(\varphi(s_1)) \neq 0$. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be natural coordinates for q near $\varphi(s_1)$ and $\varphi(s_2)$; in these coordinates $q = dz_1^2 = dz_2^2$. Then the map $\varphi \circ f \circ \varphi^{-1}$ is written

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \pm \begin{pmatrix} \lambda x \\ y/\lambda \end{pmatrix}, \quad \text{where } \lambda = \sqrt{K}. \tag{8.1.1}$$

The number λ is called the *stretch factor* of f, also known as the *dilatation factor* or *expansion factor*; see Figure 8.1.1. Note that in these coordinates, λ and $1/\lambda$ are the eigenvalues of the derivative of $\varphi \circ f \circ \varphi^{-1}$. $\triangle$

Proposition 8.1.2 *A pseudo-Anosov homeomorphism and a reducible homeomorphism cannot be homotopic.*

Let q be a quadratic differential on a Riemann surface. Recall from Section 5.3 in Volume 1 (equation 5.3.2) that in a neighborhood of any point where $q \neq 0$, there exists a *natural local coordinate* z such that $q = dz^2$; the element of length $|dz|$ in such a coordinate is denoted $|q|^{1/2}$.

PROOF Let $f: S \to S$ be pseudo-Anosov, so there exists a Riemann surface X, a quadratic differential $q \in Q(X)$, and a homeomorphism $\varphi: S \to X$ such that if $g := \varphi \circ f \circ \varphi^{-1}$, then $g: (X, q) \to (X, q/K)$ is a Teichmüller map. We will show that g is not homotopic to a reducible homeomorphism, so that f isn't either. With the metric $|q|^{1/2}$, the image of a geodesic by a Teichmüller map is a geodesic. A closed geodesic is made up of finitely many segments, each with a slope. Suppose that a geodesic γ is mapped by g to a geodesic γ' homotopic to γ. Then either γ and γ' coincide, or together they bound a straight cylinder for the metric $|q|^{1/2}$. In either case, the slopes of the segments making up γ coincide with those making up γ'.

However, a segment of slope a is mapped by g to a segment of slope a/K. Thus the only slopes that can appear for a segment of γ' are 0 and ∞. Further, the horizontal and vertical parts of γ must be mapped to the horizontal and vertical parts of γ', which must therefore have the same lengths. This contradicts the fact that g expands horizontal lines and contracts vertical ones. $\square$

Exercise 8.1.3 Show that for a pseudo-Anosov map, every horizontal trajectory is dense. $\diamondsuit$

Theorem 8.1.4 (Classification of homeomorphisms of compact surfaces) *Let S be a compact oriented surface of genus g, and let $f: S \to S$ be an orientation-preserving homeomorphism. Then f is homotopic either to a periodic homeomorphism, or to a reducible homeomorphism, or to a pseudo-Anosov homeomorphism.*