10
Rational functions

We now come to the second of Thurston's theorems treated in this book: the topological characterization of rational functions, Theorem 10.1.14.

10.0 Introduction

The combinatorial aspect of iteration theory for rational functions concerns *postcritically finite* maps: maps whose critical points have finite orbits. Think for instance of

$$f(z) = z^2 - 1, \tag{10.0.1}$$

with critical points ∞, 0 and critical orbits $\infty \mapsto \infty$, $0 \mapsto -1 \mapsto 0$, or of

$$g(z) = z^2 + i, \tag{10.0.2}$$

with critical points ∞, 0 and critical orbits

$$\infty \mapsto \infty, \quad 0 \mapsto i \mapsto -1+i \mapsto -i \mapsto -1+i. \tag{10.0.3}$$

Theorem 10.1.14 answers essentially all questions about such maps; more precisely, it reduces them to (sometimes difficult) topological problems.

It is easy to see what a *topological postcritically finite* map should be: an orientation-preserving ramified covering map $f : S^2 \to S^2$ such that all the ramification points have finite orbits. We call such maps *Thurston mappings*.

The polynomials f and g above are Thurston mappings, but not "interesting ones" in the present context, since they are already rational functions. To find an interesting one, consider $D_\gamma \circ g$, where D_γ is the Dehn twist around a simple closed curve γ on $\mathbb{C} - \{i, -1+i, -i\}$, for instance one of the curves γ_1 or γ_2 shown in Figure 10.0.1.

Thurston's theorem answers the question: when does such a Thurston mapping "look like" a rational function? This is part of the philosophy that sometimes a "floppy" topological object has a natural rigid geometry. Here the floppy topological object is a Thurston mapping and the rigid object is a rational function.

A Thurston mapping is "floppy" in part because *a priori* S^2 carries no particular complex structure. But the main source of floppiness will become clear when we say exactly what "looks like" means: f is *not* required to be conjugate to a rational function, but only conjugate to a rational function "up to homotopy" rel the orbits of the critical points.

Why the caveat that Thurston's theorem answers "essentially" all combinatorial questions about rational maps? Although the theorem says that either f looks like a rational function or there is a purely topological obstruction, these obstructions are quite difficult to understand, and almost every successful attempt to classify the obstructions is a major theorem in its own right. Theorems 10.3.1 and 10.3.3 are examples of such theorems.

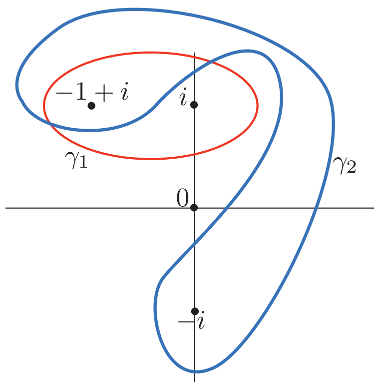

FIGURE 10.0.1 Let g be the function $g(z) = z^2 + i$ of equation 10.0.2, with postcritical set $\{\infty, i, -1+i, -i\}$. Let D_{γ_i} be the Dehn twist around a simple closed curve γ_i in the complement of the postcritical set. We can ask whether the Thurston maps $D_{\gamma_1} \circ g$ and $D_{\gamma_2} \circ g$ look like rational functions (in this case necessarily a polynomial, since ∞ is a fixed critical point). Even with Thurston's theorem, the question is difficult to answer. The solution was given by L. Bartholdi and V. Nekrashevych in 2005 in a 41-page paper [5].

This chapter has the following structure. Section 10.1 states the theorem, outlines the proof, and develops the necessary vocabulary: Thurston map, Thurston equivalence, Thurston obstruction. Sections 10.2–10.5 are devoted to one example: quadratic polynomials. We define a family of Thurston maps in Section 10.2; we analyze their Thurston obstructions in Section 10.3; in Section 10.4 we analyze the corresponding polynomials in some cases with no Thurston obstructions. In Section 10.5, we use these polynomials as "organizing centers" in a description of the Mandelbrot set. Finally, the proof of the theorem is given in Sections 10.6–10.11.

10.1 Thurston mappings

Let S^2 be the topological sphere (thus it is floppy; we are not thinking of the sphere with any particular metric or analytic structure). Let $f : S^2 \to S^2$ be an orientation-preserving ramified covering map, called a *branched map* for short. Note that a branched map is a local homeomorphism except on a discrete set; since the domain of f is compact, this set, called Crit_f, is finite. Define the *postcritical set* P_f by

$$P_f := \bigcup_{n>0} f^{\circ n}(\mathrm{Crit}_f). \qquad 10.1.1$$

Note that n is strictly greater than 0; critical points are postcritical only if they are in the orbit of a critical point, for instance by being periodic.

Definition 10.1.1 (Thurston map) A branched map $f : S^2 \to S^2$ of degree $d \geq 2$ whose postcritical set P_f is finite is called a *Thurston map*.

Exercise 10.1.2 Show that a branched map of degree $d > 0$ has $(2d-2)$ critical points, counted with multiplicity. ♢

Thurston's theorem on the topological characterization of rational functions applies only to Thurston mappings. In Section 10.2 we show how to generate many Thurston maps. All our examples will be *topological polynomials*, much easier to deal with than general Thurston maps.

Definition 10.1.3 (Topological polynomial) A Thurston map f is a *topological polynomial* if there exists a point in S^2 called ∞ such that $f^{-1}(\infty) = \{\infty\}$.

The point ∞ is then necessarily a critical point of f, and the local degree of f at ∞ is the same as the degree d of f.

Exercise 10.1.4 Let f be a topological polynomial and let $D \subset S^2 - \{\infty\}$ be a topological disc, with $\partial D \cap f(\mathrm{Crit}_f) = \emptyset$. Show that every component of $f^{-1}(D)$ is a topological disc. ♢

The next exercise is quite surprising at first. If f has degree d, then $f^{\circ k}$ has degree d^k, hence $(2d^k - 2)$ critical points, counted with multiplicity. One might expect the postcritical set to grow also, but it doesn't!

Exercise 10.1.5 Show that if $f : S^2 \to S^2$ is a Thurston map with postcritical set P_f, then for all $k \geq 1$ we have $P_f = P_{f^{\circ k}}$. ♢

Definition 10.1.6 (Thurston equivalence) Two Thurston maps f and g are *Thurston equivalent* if there exist homeomorphisms φ and φ' from S^2 to S^2 such that

1. φ and φ' coincide on P_f
2. The following diagram commutes:

$$\begin{array}{ccc} S^2 & \overset{\varphi'}{\rightarrow} & S^2 \\ \downarrow f & & \downarrow g \\ S^2 & \overset{\varphi}{\rightarrow} & S^2 \end{array} \qquad 10.1.2$$

3. φ is isotopic to φ' rel P_f

Part 3 means that there exists a path $t \mapsto \varphi_t$, $t \in [0, 1]$ in the space of homeomorphisms from the sphere to itself, with $\varphi_0 = \varphi$ and $\varphi_1 = \varphi'$, such that all φ_t coincide with φ on P_f.

Thurston equivalence is a “floppy” notion of conjugacy: two Thurston equivalent maps are “conjugate up to isotopy rel P_f”. The justification of this definition is that it makes Theorem 10.1.14 true. Since diagram 10.1.2 commutes, φ' necessarily sends Crit_f to Crit_g, hence P_f to P_g. Using φ^{-1} and $(\varphi')^{-1}$, we see that this is indeed an equivalence relation.

Before stating Theorem 10.1.14, we need to define 2-dimensional orientable *orbifolds*, which will allow us to define the orbifold of a Thurston map f.

Definition 10.1.7 (Orbifolds and their Euler characteristics)

1. A *2-dimensional orbifold* (X, ν) is an oriented surface X together with a function $\nu\colon X \to \{1, 2, \dots, \infty\}$ that assigns 1 to all but a discrete set of points.
2. The *Euler characteristic* of a compact 2-dimensional orbifold (X, ν) is
$$\chi(X, \nu) := \chi(X) - \sum_{x \in X} \left(1 - \frac{1}{\nu(x)}\right). \qquad 10.1.3$$
3. The orbifold (X, ν) is *hyperbolic* if $\chi(X, \nu) < 0$.

This definition is not transparent. We give a more conceptual presentation in Chapter 11 in Volume 3. Note that a discrete subset of a compact space is finite, and only the finitely many points $x \in X$ satisfying $\nu(x) > 1$ contribute to the sum in equation 10.1.3.

In this chapter we are concerned only with the case $X = S^2$, so that
$$\chi(S^2, \nu) = 2 - \sum_{x \in S^2} \left(1 - \frac{1}{\nu(x)}\right). \qquad 10.1.4$$

Note that if $\nu(x) > 1$, then $1 - \frac{1}{\nu(x)} \geq \frac{1}{2}$. Thus if the cardinality of the set of points where $\nu(x) > 1$ is bigger than 4, the orbifold (S^2, ν) is hyperbolic, so almost all examples are hyperbolic.

To define the orbifold of a Thurston mapping, we will need the following proposition. We denote by $\deg_y f$ the local degree of f at y.

Proposition 10.1.8 *Let $f: S^2 \to S^2$ be a Thurston map with post-critical set P_f. There exists a smallest function $\nu_f : S^2 \to \mathbb{N} \cup \{\infty\}$ such that*

1. *$\nu_f(x) = 1$ if $x \notin P_f$.*
2. *For all $x \in S^2$ and all $y \in f^{-1}(\{x\})$, the “weight” $\nu_f(x)$ is a multiple of $(\deg_y f)(\nu_f(y))$.*