Appendix C8

Thurston maps with nonhyperbolic orbifolds

In this appendix we solve two problems. We find all Thurston maps with nonhyperbolic orbifolds, and we discover which are equivalent to rational functions.

Recall from Definition 10.1.7 that a *2-dimensional orbifold* (X,ν) is an oriented surface X together with a function $\nu : X \to \{1,2,\dots,\infty\}$ that assigns 1 to all but a discrete set of points. A map $f : (S,\nu) \to S',\nu')$ is an *orbifold morphism* if for all $x \in S$, the number $\nu'(f(x))$ is a multiple of $\nu(x) \deg_x f$ (infinity divides only itself, and anything divides infinity). An orbifold morphism is a *finite covering map of orbifolds* if it is proper and $\nu'(f(x)) = \nu(x) \deg_x f$ for all $x \in S$.

Let $f : S^2 \to S^2$ be a Thurston map with orbifold $O_f = (S^2, \nu_f)$ (Definition 10.1.9). The Euler characteristic of an orbifold is defined in equation 10.1.3; an orbifold is *hyperbolic* if its Euler characteristic is < 0. The weights ν_f are defined to be the minimal integers (or ∞) so that $f : (S^2,\nu_f) \to (S^2,\nu_f)$ is a morphism of orbifolds (see Proposition 10.1.8). The map f is of course proper, but it is not a covering map when O_f is hyperbolic: there are points not in in $P_f \cup \mathrm{Crit}_f$ that map to P_f (see Proposition 10.7.5).

Proposition C8.1

1. *If* $f : S^2 \to S^2$ *is a Thurston mapping of degree* $d \geq 2$*, then* $\chi(O_f) \leq 0$.
2. *If* $\chi(O_f) = 0$*, then* $f : O_f \to O_f$ *is a finite covering map of orbifolds.*

Proof 1. Let $f : S^2 \to S^2$ be a Thurston mapping of degree $d \geq 2$, and let $\widetilde{O}_f := (S^2, \widetilde{\nu}_f)$ where

$$\widetilde{\nu}_f(x) = \frac{\nu_f(f(x))}{\deg_x f}. \tag{C8.1}$$

Then $\widetilde{\nu}_f \geq \nu_f$ everywhere (see Proposition 10.1.8), so

$$\chi(\widetilde{O}_f) \leq \chi(O_f). \tag{C8.2}$$