Appendix C7
Examples of Thurston pullback maps

Thurston pullback maps $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ were until recently pretty mysterious objects (except the non-hyperbolic ones examined in Appendix C8, which are highly atypical). But work by L. Bartholdi, X. Buff, A. Epstein, A. Koch, C. McMullen, V. Nekrashevych, K. Pilgrim, and others has made their behavior much more transparent. In this section we will show some examples, which we hope are fairly typical.

Recall that $\overline{\sigma}_f$ is defined in equation 10.9.5. Lemma 10.9.3 says that there exists a tower of covering maps p, p_1, and p_2 such that the diagram

$$\begin{array}{ccc} \mathcal{T}_f & \xrightarrow{\sigma_f} & \mathcal{T}_f \\ \downarrow p_1 & & \\ \text{Moduli}'_f & & \Big\downarrow p \\ \downarrow p_2 & \searrow \ \overline{\sigma}_f & \\ \text{Moduli}_f & & \text{Moduli}_f \ . \end{array}$$

commutes, with the projection p_2 finite.

The key point is that both Moduli_f and Moduli'_f are algebraic varieties, hence susceptible to computation. In the special case that we will investigate, that of $\sigma_{\tilde{f}_\theta}$ for $\theta = 1/7$, the computation reveals something that isn't always true, but is true a lot more often that one might expect: the map $\overline{\sigma}_f$ is injective, and its image is the complement of an algebraic subvariety $Z \subset \text{Moduli}_f$. Thus there is a map

$$p_2 \circ (\overline{\sigma}_f)^{-1} : \text{Moduli}_f - Z \to \text{Moduli}_f, \qquad C7.1$$

a sort of inverse of σ_f on the level of the moduli space.

Maps on moduli space

It is natural to wonder whether σ_f induces a map of moduli spaces: is there a map $\widehat{\sigma}_f : \text{Moduli}_f \to \text{Moduli}_f$ such that the diagram

$$\begin{array}{ccc} \mathcal{T}_f & \xrightarrow{\sigma_f} & \mathcal{T}_f \\ p \downarrow & & \downarrow p \\ \text{Moduli}_f & \xrightarrow{\widehat{\sigma}_f} & \text{Moduli}_f \end{array} \qquad C7.2$$

commutes? Recall that $p : \mathcal{T}_f \to \text{Moduli}_f$ is a universal covering map, and the associated covering group is the *pure* mapping class group, which consists of orientation-preserving homeomorphisms $h : S^2 \to S^2$ fixing the