Appendix C6

Linearizing at irrationally indifferent fixed points

In Chapter 9 we treated linearization of analytic functions at repelling fixed points and at linearly attracting fixed points. We also discussed the case of rationally indifferent fixed points. In this appendix we give a bare-bones treatment of a very important topic, the linearization of analytic functions f with irrationally indifferent fixed points.

If you put the fixed point at the origin, such a function can be written as the convergent power series

$$f(z) = \lambda z + a_2 z^2 + \cdots, \text{ with } f'(0) = \lambda = e^{2\pi i\theta}, \ \theta \in (\mathbb{R} - \mathbb{Q})/\mathbb{Z}. \quad C6.1$$

Figure C6.1 illustrates what it means to linearize a function with an irrationally indifferent fixed point.

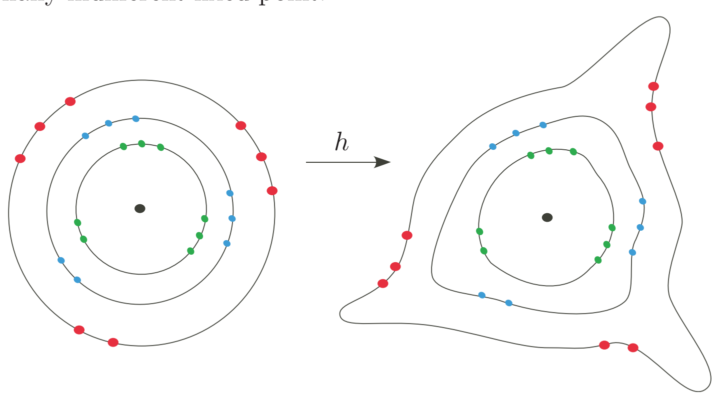

FIGURE C6.1 Multiplication by $\lambda = e^{2\pi i\theta}$ is simply rotation by θ turns. When θ is irrational, orbits under multiplication by λ are dense in circles centered at the origin, and no point other than the origin is periodic. If h is a linearizing chart for f at the origin, i.e., if $h(\lambda z) = f(h(z))$, then h maps orbits under multiplication by λ to orbits under f; such orbits will be dense in the images of these circles, i.e., in real analytic closed curves. In particular, orbits under f near the origin will be defined forever, and bounded away from 0 and infinity.

REMARK Orbits bounded away from 0 and infinity should remind you of one of the great puzzles of physics: *is the solar system stable?* Will