Appendix C5
Totally real stretch factors

Here we sketch a result of Ursula Hamenstädt [37]: "most" stretch factors λ of pseudo-Anosov homeomorphisms are totally real: they are algebraic integers and all their Galois conjugates are real. We saw in Chapter 8 that all stretch factors are algebraic integers; here we show that the Galois conjugates of "most" stretch factors are real. The argument uses a result of Artur Avila and Marcelo Viana [4]: all exponents of a particular Lyapunov spectrum of the Teichmüller flow on moduli space are simple. The discussion uses sheaf cohomology; see Appendix A7 of Volume 1.

Quadratic differentials, holomorphic 1-forms, and the Teichmüller flow

Let S be a compact surface of genus g, and let $\mathcal{Q}_S$ be the bundle over Teichmüller space $\mathcal{T}_S$ whose fiber over $\tau \in \mathcal{T}_S$, represented by $\varphi : S \to X$, is the vector space $Q(X)$, the space of holomorphic quadratic differentials on X. The bundle $\mathcal{Q}_S$ is also the cotangent bundle to $\mathcal{T}_S$. (This is a special case of Proposition 6.6.2.) It is a complex manifold of dimension $6g-6$.

It will be more convenient to work with pseudo-Anosov homeomorphisms such that the associated quadratic differential is the square of an Abelian differential, i.e., a holomorphic 1-form. (The arguments go through in the more general case by taking an appropriate double cover, but the proofs are more elaborate without introducing new ideas.) Thus, let $\mathcal{C}_S$ be the vector bundle over $\mathcal{T}_S$ whose fiber above τ is $\Omega_X(X)$, the space of holomorphic 1-forms on X. This vector bundle $\mathcal{C}_S$ (the total space, not just the fiber) is a complex manifold of dimension $4g-3$, and the map $\omega \mapsto \omega^2$ makes $\mathcal{C}_S$ into a double cover of a bundle of cones embedded in $\mathcal{Q}_S$. The elements of $\mathcal{C}_S$ are often called *translation surfaces*.

The *Teichmüller flow* $\Phi : \mathbb{R} \times \mathcal{Q}_S \to \mathcal{Q}_S$ is best described in terms of natural coordinates: if $\tau \in \mathcal{T}_S$ is represented by $\varphi : S \to X$ and $q \in Q(X)$ is written $q = dz^2 = (dx + i\,dy)^2$ in a local coordinate $z = x + iy$, then the point $\Phi(t,(\tau,q))$ is the point (τ', q'), where

1. τ' is represented by $\varphi : S \to X'$, where X' has underlying surface X and $z' = x' + iy' = e^t x + ie^{-t} y$ is an analytic coordinate
2. $q' = (dz')^2 = (dx' + idy')^2 = (e^t dx + ie^{-t} dy)^2$

This flow preserves $\mathcal{C}_S$, and in the notation above, $\Phi(t,(\tau,\omega)) = (\tau', \omega')$, where

1. $\omega \in \Omega_X(X)$ is written in a natural coordinate $\omega = dz = dx + idy$