Appendix C3
Homotopy implies isotopy

In topology, we are constantly up against an irritating distinction between homotopy and isotopy. The object of Theorem C3.1 is to lay this ghost for once and for all. We will use the results and notation of Section 3.6.

Theorem C3.1 *Let S be a second-countable connected orientable surface. If two homeomorphisms $f_1, f_2 : S \to S$ are homotopic, then they are isotopic.*

Proof We will prove this when S carries a complete hyperbolic structure and admits a decomposition by a multicurve Γ into geodesic trousers and half-annuli; thus every $\gamma \in \Gamma$ is the geodesic in its homotopy class. The special cases where S is a sphere, a torus, a disc, or an annulus are a bit different. Exercise C3.3 sketches how to modify the proof in those cases.

We are also assuming that S has no boundary. Exercise C3.4 sketches how to adapt the proof when S has a boundary consisting of circles.

We will further make the (less innocent) assumption that the set Z_Γ of points of curves $\gamma \in \Gamma$ is closed. It follows from the discussion in the second paragraph after Figure 3.6.3 that Z_Γ being closed is equivalent to requiring that all components of the ideal boundary $I(S)$ be circles; none are homeomorphic to $\mathbb{R}$. This hypothesis on S is satisfied by all surfaces S such that $H_1(S; \mathbb{Z})$ has finite rank, and by many more. Exercise C3.5 sketches how to adapt the proof if Z_Γ is not closed.

Theorem C3.1 is clearly equivalent to proving that if $f : S \to S$ is a homeomorphism homotopic to the identity, then it is isotopic to the identity. By Proposition 6.4.9, this is true when S has a conformal structure for which f is quasiconformal, so here the emphasis will be on the case where we have no smoothness hypotheses whatsoever on f. Further, in Lemma A2.3 (in the appendix of Volume 1), we proved a result about smooth curves in S; we will revisit this result here without smoothness conditions.

Let Δ be the set of geodesics joining one boundary component of a trouser to another boundary component of the same trouser; there are three such geodesic segments δ', δ'', δ''' in each trouser of $S - Z_\Gamma$. Let Z_Δ be the set of points of elements $\delta \in \Delta$.

We are really doing topology, and can choose our hyperbolic surface S within its topological class. Recall that Fenchel-Nielsen coordinates are discussed in Section 7.6 in Volume 1. It will be convenient to choose all the Fenchel-Nielsen twists to be 0, so that geodesic segments joining curves of Γ within trousers and in neighboring trousers continue each other.