Appendix C2

The Alexander trick

The proof we give of Proposition C2.1 is due to the American topologist James Alexander (1888–1971); it is called the *Alexander trick*.

Proposition C2.1 (Alexander trick)

1. *Every orientation-preserving homeomorphism* $f : S^{n-1} \to S^{n-1}$ *extends to a homeomorphism* $\widetilde{f} : \mathbf{D}^n \to \mathbf{D}^n$.
2. *Any two such extensions* $\widetilde{f}_1, \widetilde{f}_2$ *are isotopic through a family of homeomorphisms* $\tilde{f}$ *such that* $\tilde{f}|_{S^{n-1}} = f$ *for all* $t \in [0, 1]$.

Proof 1. Let $f : S^{n-1} \to S^{n-1}$ be a homeomorphism. Define the *radial extension* by

$$\widetilde{f}(\mathbf{x}) := \begin{cases} |\mathbf{x}|\, f\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) & \text{if } \mathbf{x} \in \mathbf{D}^n - \{\mathbf{0}\} \\ \mathbf{0} & \text{if } \mathbf{x} = \mathbf{0}. \end{cases} \qquad C2.1$$

Clearly this map is a homeomorphism $\mathbf{D}^n \to \mathbf{D}^n$ extending f.

2. Let $\widetilde{g} : \mathbf{D}^n \to \mathbf{D}^n$ be another homeomorphism extending f. Consider the one-parameter family of homeomorphisms F_t defined for $0 < t \leq 1$ by

$$F_t(\mathbf{x}) := \begin{cases} \widetilde{f}(\mathbf{x}) & \text{if } |\mathbf{x}| \geq t \\ t\widetilde{g}\,(\mathbf{x}/t) & \text{if } |\mathbf{x}| \leq t \end{cases} \qquad C2.2$$

(If $|\mathbf{x}| = t$, then $t\widetilde{g}(\mathbf{x}/t) = tf(\mathbf{x}/t) = |\mathbf{x}|f(\mathbf{x}/|\mathbf{x}|) = \widetilde{f}(\mathbf{x})$.) Then F_t converges to $\widetilde{f}$ uniformly on $\mathbf{D}^n$ as $t \to 0$. Clearly $F_1(\mathbf{x}) = \widetilde{g}(\mathbf{x})$. This constructs an isotopy between $\widetilde{g}$ and $\widetilde{f}$, showing that any extension is isotopic to the radial extension, hence all extensions are isotopic to each other. □

Remarks 1. Even if f is a diffeomorphism, the radial extension is emphatically not differentiable at 0.

2. The map $f \mapsto \widetilde{f}$ is a group homomorphism

$$\text{Homeo}(S^{n-1}) \to \text{Homeo}(\mathbf{D}^n) \qquad C2.3$$

that is a section of the restriction homomorphism

$$\text{Homeo}(\mathbf{D}^n) \to \text{Homeo}(S^{n-1}).$$

3. If we replace "homeomorphism" by "diffeomorphism", the result remains true for $n = 1$ and $n = 2$, but it is false in higher dimensions. For $n = 2$ the result is very hard [18]. △