Theorem 2.10.7 is illustrated by Figure 2.10.6.

Note that we could replace the length of the derivatives by their norm (Definition 2.9.6) to get a slightly stronger theorem.

On first reading, skip the last sentence about the ball with radius $R_1$. It is a minor point.

The ball $V$ gives a lower bound for the domain of $g$: the actual domain may be bigger. But there are cases where the largest $R$ satisfying the conditions of Theorem 2.10.7 is optimal. We invite you to check that if $f(x) = (x - 1)^2$, with $x_0 = 0$, so that $y_0 = 1$, then the largest $R$ satisfying equation 2.10.11 is $R = 1$. Thus the interval $V = (0, 2)$ is the largest interval centered at 1 on which an inverse can be defined. Indeed, since the function $g$ is $g(y) = 1 - \sqrt{y}$, any value of $y$ smaller than 0 is not in the domain of $g$.

**Theorem 2.10.7 (The inverse function theorem).** Let $W \subset \mathbb{R}^m$ be an open neighborhood of $x_0$, and let $f : W \rightarrow \mathbb{R}^m$ be a continuously differentiable function. Set $y_0 = f(x_0)$.

If the derivative $[Df(x_0)]$ is invertible, then $f$ is invertible in some neighborhood of $y_0$, and the inverse is differentiable.

To quantify this statement, we will specify the radius $R$ of a ball $V$ centered at $y_0$, in which the inverse function is defined. First simplify notation by setting $L = [Df(x_0)]$. Now find $R > 0$ satisfying the following conditions:

1. The ball $W_0$ of radius $2R|L^{-1}|$ and centered at $x_0$ is contained in $W$.
2. In $W_0$, the derivative of $f$ satisfies the Lipschitz condition

$$||Df(u)| - |Df(v)|| \leq \frac{1}{2R|L^{-1}|^2} |u - v|.$$  \hspace{1cm} 2.10.11

Set $V = BR(y_0)$. Then

1. There exists a unique continuously differentiable mapping $g : V \rightarrow W_0$ such that $g(y_0) = x_0$ and $f(g(y)) = y$ for all $y \in V$.  \hspace{1cm} 2.10.12

Since the derivative of the identity is the identity, by the chain rule, the derivative of $g$ is $[Dg(y)] = [Df(g(y))]^{-1}$.

2. The image of $g$ contains the ball of radius $R_1$ around $x_0$, where

$$R_1 = R|L^{-1}|^2 \left( \sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}} - |L| \right).$$  \hspace{1cm} 2.10.13

![Figure 2.10.6.](image-url)

**Figure 2.10.6.** The function $f : W \rightarrow \mathbb{R}^m$ maps every point in $g(V)$ to a point in $V$; in particular, it sends $x_0$ to $f(x_0) = y_0$. Its inverse function $g : V \rightarrow W_0$ undoes that mapping. The image $g(V)$ of $g$ may be smaller than its codomain $W_0$. The ball $BR_1(x_0)$ is guaranteed to be inside $g(V)$; it quantifies “locally invertible”.