It seems easier to remove b next, writing

$$-\frac{11}{12}(a+b)^2 + \frac{11}{12}a^2 - \frac{56}{45}a^2.$$

We are left with $\frac{-59}{180}a^2$. So the quadratic form is

$$\left(c + \frac{a}{3} + \frac{b}{2}\right)^2 - \frac{11}{12}(a+b)^2 - \frac{59}{180}a^2, \quad \text{with signature } (1,2).$$

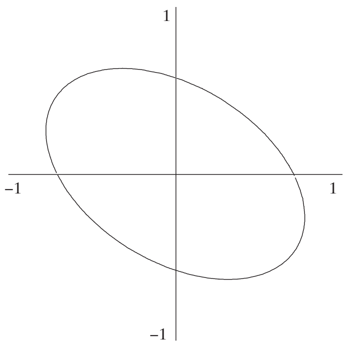

FIGURE 1 FOR SOLUTION 3.5.17, part a. The ellipse of equation $2x^2 + 2xy + 3y^2 = 1$.

3.5.17 a. The quadratic form corresponding to this matrix is

$$[x\ y]\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 2xy + 3y^2.$$

Completing squares gives $2x^2 + 2xy + 3y^2 = 2(x + y/2)^2 + 5y^2/2$, so the quadratic form has signature $(2,0)$, and the curve of equation

$$2x^2 + 2xy + 3y^2 = 1$$

is an ellipse, drawn in the margin (Figure 1).

b. The quadratic form corresponding to the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{is} \quad x^2 + y^2 + z^2 - 2xy - 2yz.$$

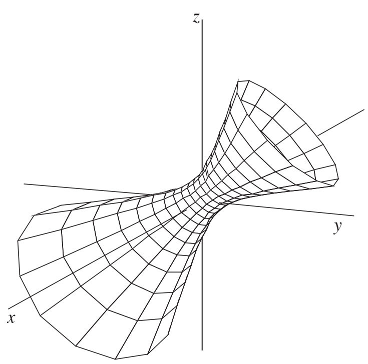

FIGURE 2 FOR SOLUTION 3.5.17 The hyperboloid of part b.

This can be written

$$x^2 + y^2 + z^2 - 2xy - 2yz = (x+y+z)^2 + \frac{1}{2}(y-z)^2 - \frac{1}{2}(y+z)^2,$$

which has signature $(2,1)$; the surface of equation

$$x^2 + y^2 + z^2 - 2xy - 2yz = 1$$

is a hyperboloid of one sheet (Figure 2).

We will specify how the surface is parametrized. A parametrization of the surface of equation $u^2 + v^2 - w^2 = 1$ is given by setting

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh s \cos t \\ \cosh s \sin t \\ \sinh s \end{pmatrix}, \quad s \in \mathbb{R}, \quad 0 \le v \le 2\pi.$$

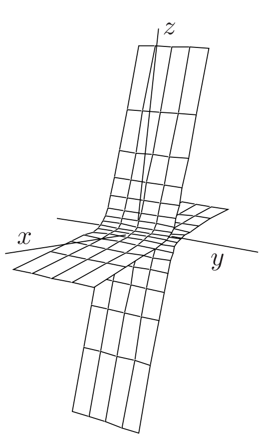

FIGURE 3 FOR SOLUTION 3.5.17, part c. The hyperbolic cylinder.

Thus if we solve the equations

$$x + y + z = \cosh s \cos t$$
$$\frac{y-z}{\sqrt{2}} = \cosh s \sin t$$
$$\frac{y+z}{\sqrt{2}} = \sinh s$$

for x, y, and z in terms of s and t, we will have parametrized the surface.